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Coherent phase spaces. Semiclassical semantics

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Abstract

The category of coherent phase spaces introduced by the author is a refinement of the symplectic “category” of A. Weinstein. This category is $*$-autonomous and thus provides a denotational model for Multiplicative Linear Logic. Coherent phase spaces are symplectic manifolds equipped with a certain extra structure of “coherence”. They may be thought of as “infinitesimal” analogues of familiar coherent spaces of Linear Logic. The role of cliques is played by Lagrangian submanifolds of ambient spaces. Physically, a symplectic manifold is the phase space of a classical dynamical system, and a Lagrangian submanifold is a phase of a short-wave oscillation. Typically, Lagrangian submanifolds represent such objects as short-wave approximations of wave functions (semiclassical states) in asymptotic quantization and wave fronts in geometrical optics. The coherent phase space semantics was motivated to a large extent by methods of geometric and asymptotic quantization and suggests some interesting intuitions on Linear Logic. In particular Lagrangian submanifold-cliques of types A and $A^{\perp}$ can be interpreted as semiclassical limits of eigenstates of respectively position and momentum observables. These observables being canonically conjugate cannot be measured simultaneously, which corresponds to the idea that a formula A and its negation $A^{\perp}$ cannot both simultaneously have proofs (models).

We show that the coherent phase space semantics of Linear Logic enjoys several completeness properties in general much stronger than the usual full completeness with respect to the class of dinatural transformations. These properties of completeness in conjunction with a quite natural (quasi)-physical meaning make the coherent phase space semantics an interesting object of investigation.

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1. Introduction

1.1. Semiclassical semantics

From the very emergence of Linear Logic there has been a feeling that there should be some analogies with physics. It seems to be a common idea that some (quasi-)physical intuitions might be helpful for understanding Linear Logic (the ultimate hope being that this will work in the opposite direction as well, i.e. that Linear Logic intuitions will help to understand quantum theory). Among the works apparently inspired by this idea let us mention [4,7,12].

The coherent phase space semantics of Multiplicative Linear Logic can also be seen as an attempt to apply this idea more or less literally. However, unlike the works cited above, in this work we try to derive some intuition from geometric rather than "algebraic" (i.e. concerned with Hilbert spaces and operator algebras) aspects of the quantum theory. More precisely our semantics of Linear Logic is inspired by the ideas of geometric quantization developed by Kirillov [15], Konstant [16] and Souriau [23] and semiclassical (asymptotic) approximation due mainly to Maslov [14].

The title "phase spaces" is of course a joke; it does not seem to refer in any way to the standard phase space semantics. Coherent phase spaces are symplectic manifolds equipped with a certain extra structure of "coherence", that is with a *field of contact cones*, which is a subset of the tangent bundle closed under scalar multiplication. These may be thought of as "infinitesimal" analogues of familiar coherent spaces of Linear Logic. The role of cliques is played by Lagrangian submanifolds of the ambient spaces, which are tangent to corresponding fields. The "Lagrangianness" property in this context may be thought of as an infinitesimal analogue of totality. Physically, a symplectic manifold is the phase space of a classical dynamical system. Lagrangian submanifolds of the phase space represent such objects as short-wave approximations to wave functions (semiclassical states) in asymptotic quantization and wave fronts in geometrical optics. The main physical meaning of a Lagrangian submanifold is that of the best possible localization of a quantum particle in the classical phase space. (Typically, a measurement of all spacial coordinates of a particle localizes the particle at the corresponding submanifold of the phase space. This localization cannot be improved due to the Heisenberg uncertainty principle.) It was precisely this "semi-classical" interpretation which led us to the coherent phase space semantics. Nevertheless, as we shall try to show below, this semantics makes sense from a purely mathematical point of view as well.

Coherent phase spaces were introduced in [22]. In this paper we investigate completeness questions of the coherent phase space semantics.

This question is rather subtle since we are interested in modeling proofs and not provability only. Today we know plenty of models of various fragments of Linear Logic: especially interesting models being based on structures which arise from general mathematical practice, such as topological vector spaces, C^*-algebras, games etc. There are models known to be complete in some sense such as coherent spaces [24], games [1], topological vector spaces with a continuous action of the additive group of integers [5]. Known completeness theorems (full completeness theorems) usually state the completeness of the interpretation with respect to the class of dinatural transformations in the chosen category. However the class of dinatural transformations may seem rather abstract; it appears somehow that dinatural transformations do not reflect too much the specific structure of a concrete category under consideration. The category of coherent spaces known to be fully complete possesses quite a lot of transformations which while not being dinatural are fairly "natural" in the informal sense of this word. (Take a clique in $A \multimap A \otimes A$ of the form $\{(x, x, x)\}$.) One may argue that various models mentioned above although being complete in some formal sense fail to capture adequately the structure of Linear Logic in a more informal sense.

It turns out that coherent phase spaces provide a model for Multiplicative Linear Logic (with the Mix rule) which is not only fully complete in the sense of completeness with respect to dinatural transformations, but reveals a remarkable flexibility with respect to definitions of completeness. Varying the criterion of completeness we obtain below three different completeness theorems.

We assume that the reader is familiar with elementary notions of differential geometry. For general references on symplectic geometry we use [19]. The main sources for geometric quantization and related ideas are [27,13]; a short introduction into the subject is [3].

The word "smooth" always means infinitely smooth and all manifolds, submanifolds, functions, vector fields etc are assumed to be smooth unless otherwise stated. A vector v tangent or cotangent to a manifold at the point q is usually written with the corresponding subscript, i.e. as v_q or as a pair (q, v_q). We use both upper and lower indices in coordinate formulas; upper indices are usually for tangent vectors and lower indices are for cotangent vectors as is standard in differential geometry and physics. The differential of a map f (the tangent map) is denoted by Tf; the differential of f at the point q is denoted by $T_q f$.

1.2. Phase spaces

In the simplest case the phase space of a dynamical system is the vector space $\mathbf{R}^{2n}$ coordinatized by $2n$-tuples

$$(p_1, \ldots, p_n,\ q^1, \ldots, q^n) \tag{1}$$

(momenta and positions). Here the space of positions $\{(q^1, \ldots, q^n) \mid q \in \mathbf{R}^n\}$ is the *configuration space* $Q = \mathbf{R}^n$ and the phase space $\mathbf{R}^{2n} = T^*Q$ (the cotangent bundle) is the space of all possible kinematic states of motion.

Dynamic is governed by the Poisson bracket $\{.,.\}$ defined on functions on the phase space:

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right).$$

Coordinate functions (1) satisfy

$$\{q^i, q^j\} = \{p_i, p_j\} = 0,\ \{p_j, q^i\} = \delta^i_j. \tag{2}$$

The pairs (p_i, q^i) of coordinate functions are called *canonically conjugate*.

Any coordinates on $\mathbf{R}^{2n}$ satisfying relations (2) are called *canonical*, and the laws of dynamic are identical in any canonical coordinate system. Given a Hamiltonian (energy function) $H \in C^\infty(\mathbf{R}^{2n})$ the evolution of any observable $f \in C^\infty(\mathbf{R}^{2n})$ is given by the *Hamilton equations*

$$\frac{\partial f}{\partial t} = \{f, H\}. \tag{3}$$

In more general cases the phase space is no longer a vector space but rather a *symplectic manifold*. In particular no global separation of coordinates into "positions" and "momenta" is guaranteed to exist. However the Poisson bracket is always present, and locally the situation is always similar. Canonical coordinates may be chosen locally and dynamical equations take the same form (3) as in $\mathbf{R}^{2n}$.

The importance of Hamiltonian formulation of classical mechanics became evident after the discovery of quantum mechanics. When quantum phenomena are taken into account observables can no longer be interpreted as functions on the phase space; it is well known that quantum observables form a non-commutative algebra. The Poisson bracket on the classical phase space is precisely the bridge between classical and quantum formalisms. If a hat $\hat{(.)}$ denotes the "quantization map", i.e. the map sending a classical observable f, which is just a function on the phase space, to the corresponding quantum observable $\hat{f}$ then the following identity must hold:

$$\widehat{\{f, g\}} = \frac{i}{\hbar}[\hat{f}, \hat{g}] + O(\hbar^2).$$

Here $\hbar$ is the Planck constant and square brackets denote the commutator of operators.

Non-commutativity of quantum observables is a mathematical expression for quantum uncertainty. The *Heisenberg uncertainty principle* says that two observables whose commutator does not vanish cannot be measured simultaneously. The original formulation of this principle is concerned with observables $\hat{q}$, $\hat{p}$ of position and momentum respectively, which, as follows from relations (2), do not commute. The uncertainty principle in its most popular version says that the uncertainty in position is inverse proportional to the uncertainty in momentum. Due to this uncertainty quantum states cannot be identified with points of the phase space; rather they are "spread over" the phase space. A quantum particle has no well-defined coordinates in the phase space, but it has a well-defined probability of being observed at a given point. In particular since momenta and positions cannot be simultaneously measured only half of coordinate functions can be attributed definite values at a given state; moreover these coordinates should be such that all Poisson brackets of corresponding functions pairwise vanish. It may still make sense to talk about the localization of a quantum particle in the classical phase space, but this localization occurs not at a point but at a submanifold, namely at the level set of half of coordinate functions; moreover all Poisson brackets between these coordinates should pairwise vanish. Such submanifolds are called *Lagrangian*. Lagrangian submanifolds are basic entities in our model and they deserve some discussion.

1.3. *Lagrangian submanifolds*

These are submanifolds of the phase space which are locally given as level sets of maximal collections of Poisson-commuting functions, i.e. functions whose Poisson brackets pairwise vanish. In fact it is not hard to see from relations (2) that locally around any point of $\mathbf{R}^{2n}$ there exist no more than n functionally independent Poisson commuting functions. (Natural examples of such "maximal commutative" tuples are the tuples $(q^1, \ldots, q^n)$ and $(p_1, \ldots, p_n)$.) Thus a Lagrangian submanifold of the phase space can be described as the level-set of a Poisson commuting n-tuple.

Lagrangian submanifolds play an omnipresent role in symplectic geometry. Analytically, Lagrangian submanifolds of the cotangent bundle $T^*\mathbf{R}^n \cong \mathbf{R}^{2n}$ of $\mathbf{R}^n$ are just generalized solutions of first order PDEs. (In fact, symplectic geometry is a geometric justification for the Hamilton–Jacobi theory.) In geometric optics Lagrangian submanifolds are wave fronts. In semi-classical approximation Lagrangian submanifolds are short-wave limits of quantum-mechanical states (semi-classical states). In short their role is summarized by the "symplectic creed" of A. Weinstein [25]:

EVERYTHING IS A LAGRANGIAN SUBMANIFOLD. (4)

Let us discuss various meanings of Lagrangian submanifolds in a little more detail.

1.3.1. Lagrangian submanifolds as generalized solutions

Given a function $u \in C^\infty(\mathbf{R}^n)$ the graph $\{(\frac{\partial u}{\partial q^1}(q), \ldots, \frac{\partial u}{\partial q^n}(q), q^1, \ldots, q^n) \mid q \in \mathbf{R}^n\}$ of du is a Lagrangian submanifold of $\mathbf{R}^{2n}$. A PDE of the form

$$F\left(q^1, \ldots, q^n, \frac{\partial u}{\partial q^1}, \ldots, \frac{\partial u}{\partial q^n}\right) = 0 \tag{5}$$

determines in the obvious way a submanifold C of $\mathbf{R}^{2n}$. The problem of solving (5) may be restated then in geometric terms: find a Lagrangian submanifold σ of $\mathbf{R}^{2n}$ lying in C. If initial conditions are given this can be done by solving the Hamilton equations (which are ordinary differential equations) for the Hamiltonian F. Under favourable circumstances the resulting submanifold σ is indeed the graph of a differential form, which (at least locally) can be integrated to give a particular solution u of (5). This however can be the case if and only if the projection

$$\pi : \mathbf{R}^{2n} \to \mathbf{R}^n,\ (p, q) \mapsto q \tag{6}$$

locally projects σ diffeomorphically onto $\mathbf{R}^n$ or, equivalently, at each of its points the submanifold σ is transversal to the fibers of π; otherwise the submanifold σ becomes vertical and cannot be a graph. Since in many cases this transversality condition for σ does not hold whereas the submanifold σ itself always can be found it makes a lot of sense to speak about σ as a *generalized solution* of (5). Thus fibers of π determine a certain specification which should be enjoyed by true solutions. In other terms fibers of π induce the "coherence" structure on the ambient phase space: a Lagrangian submanifold σ is coherent if the tangent bundle $T\sigma$ of σ has no intersections with vectors tangent to fibers of π (vertical vectors). This is in our opinion a very natural example where the introduction of a "coherence" structure on the phase space makes sense.

1.3.2. Lagrangian submanifolds as quantum points

A quantization of a classical system associated to the phase space M should assign to the algebra $C^\infty(M)$ of classical observables a non-commutative operator algebra of quantum observables and to each individual classical observable f a quantum observable $\hat{f}$.

As we have already discussed above a quantum particle cannot be localized at a point of the phase space as this would imply that values of both all position and all momentum coordinates are known. At best the particle may be localized by either its position or its momentum. More generally one may speak about localization of the particle at a Lagrangian submanifold of the phase space. That is the reason why Lagrangian submanifolds of the phase space are sometimes called “quantum points”.

The starting point of geometric quantization is a choice of a *polarization*, i.e. a foliation of the (classical) phase space into Lagrangian submanifolds, quantum states being defined as functions on the set of leaves of the polarization. In particular choosing between the polarizations of $\mathbf{R}^{2n}$ given by the level sets of position and of momentum coordinates one obtains the momentum or the position representation respectively. A quantum point in the position representation is then a vertical submanifold of $\mathbf{R}^{2n}$, that is a fiber of the projection π in (6). Such a submanifold cannot be interpreted as a function on the configuration (position) space $\mathbf{R}^n$, but has the meaning of a *generalized function*. A fiber of π corresponds to a δ-like distribution in the position space (a “function”, which is zero everywhere except one point, where it “attains a so large infinity” that the integral of the δ-function over the whole space equals 1). The position coordinates of a particle localized at such quantum point have definite values whereas its momentum coordinates can take all values with equal probability. Similarly a quantum point in the momentum representation is a δ-like distribution in the momentum space, that is a submanifold of the form $\{(p_0, q) | \, q \in \mathbf{R}^n\}$. Two quantum points corresponding to the position and the momentum representations correspond to two transversal Lagrangian submanifolds. (Of course one can choose infinitely many representations other than position or momentum.)

We interpret formulas as symplectic manifolds and proofs as Lagrangian submanifolds. The role of the coherence relation is played by a *field of contact cones* — a subset of the tangent bundle of the phase space, which is closed under multiplication by scalars. In particular the negation $A^\perp$ of a formula A is interpreted by the coherent phase space $[[A^\perp]]$ which is the same object as $[[A]]$ as a manifold, but whose structure of a coherent phase space is determined by the field of contact cones complementary to that of $[[A]]$. The considerations above give rise to an intuition that dual LL formulas A and $A^\perp$ have the meaning of canonically conjugate observables, which, due to the Heisenberg principle, cannot be measured simultaneously. This corresponds to the idea that A and $A^\perp$ cannot both simultaneously have proofs (models). Typically, “semantical proofs” of A and its negation $A^\perp$ are transversal Lagrangian submanifolds of the phase space, which are tangent to complementary fields of contact cones, much like two “quantum points” corresponding to particles localized in the space of positions and the space of momenta respectively. Typically, if A is the field of directions tangent to the level-sets of position variables in $\mathbf{R}^{2n}$ then the “cliques” of A are precisely the quantum points in the position representation. On the other hand the level-sets of momentum become states of $A^\perp$.

Thus, if we interpret Lagrangian submanifolds as quantum points then our “coherence” looks like a specification for the class of states and observables, which have definite values in these states. Such a specification may be relevant in the following physical context. Any concrete experimental situation determines the class of those questions and statements about the state of the system, which can be asked or verified in the current setting, and the class of those statements which do not make sense under these circumstances.

Typically, a question about the position of a particle whose momentum is being measured has no meaning since a precise measurement of position simply will destroy the system. Different statements about the system serve to localize it in the phase space and the class of possible quantum points that the system can occupy is limited by the current experimental set-up. This class can be seen as a specific "coherence" structure on the phase space. (One should be cautious with terminology, by the way, as the word "coherence" has its own meaning in quantum mechanics.)

(We should say however that although the analogy between the Linear Logic duality $A/A^{\perp}$ and the physical duality *position*/*momentum* seems to us plausible and perhaps even exciting the correspondence between our interpretation of linear negation and the meaning of canonically conjugate coordinates in geometric models of mechanic is only approximate. The pair $A/A^{\perp}$ in our model corresponds to a partition of the tangent bundle of the phase space into two complementary subsets, whereas the pair *position*/*momentum* corresponds to a decomposition of the tangent bundle into two complementary integrable subbundles. The second structure is clearly more informative. If our analogy makes sense indeed then our model should be taken only as a first approximation.)

1.3.3. Lagrangian submanifolds as semi-classical states

The concept of a quantum point is a certain idealization; in reality quantum states are represented by square integrable functions, and δ-like distributions do not belong to this class. More realistically one can speak about a wave function, which is concentrated at a quantum point being zero outside its very small neighborhood. This idealization makes sense in the context of *semi-classical approximation*.

The word "semi-classical" is not a very precise term; loosely speaking it refers to any kind of analysis where one attempts to derive as much information about the quantum situation from the corresponding classical one as possible. A physicist's exposition of semi-classical methods can be found in [17]. Usually in semi-classical analysis one represents all quantities of interest as functions of the Planck constant $\hbar$ and computes asymptotic values modulo $\hbar^2$ as $\hbar \to 0$. In good situations such a procedure allows one to get a qualitative description of the physics under consideration. At the present stage of development of quantum theory such a qualitative approach along with its drawbacks has the advantage of being less sensitive to many analytical difficulties, see [14]. A typical example of a semi-classical description goes as follows.

Given a (non-relativistic) system whose configuration space is $\mathbf{R}^n$ and whose potential function is $V \in C^{\infty}(\mathbf{R}^n)$ the dynamics is described by the Schroedinger equation

$$-\hbar^2 \left(\frac{\partial^2}{\partial (q^1)^2} + \cdots + \frac{\partial^2}{\partial (q^n)^2} \right) u + V(q)u = i \frac{\partial u}{\partial t}, \tag{7}$$

where u is the wave function of the system.

Assume that we are interested only in the behaviour of the system modulo terms of order $\hbar^2$. We may substitute $u = A(q)e^{i\phi(q)/\hbar}$ in (7), where A does not depend on $\hbar$, and try to solve the resulting equation for A (the amplitude function) and ϕ (the phase function) modulo $\hbar^2$. Assume for simplicity we want to find stationary states (i.e. eigenfunctions of the Schroedinger operator in the left-hand side of (7)) of the system with the energy level (i.e. eigenvalue) E. Then ϕ satisfies the Hamilton–Jacobi equation of the classical system

$$\left(\frac{\partial\phi}{\partial q^1}\right)^2 + \cdots + \left(\frac{\partial\phi}{\partial q^n}\right)^2 + V(x) = E \tag{8}$$

and can be found by means of the Hamilton–Jacobi theory by solving the Hamilton equations for the classical Hamiltonian $\frac{1}{2}p^2 + V(x)$. When the phase ϕ is known the amplitude A can be determined modulo $\hbar^2$ from further analysis.

The problem is that Eq. (8) is not guaranteed to have a true solution and it is natural to consider generalized solutions in the sense discussed above in Section 1.3.1. The phase function is represented then by a Lagrangian submanifold (quantum point). A *semi-classical state* [3] is defined as a Lagrangian submanifold of the phase space plus some extra data (a half-density on this submanifold tensored with a phase function, which is in general a parallel section of a certain bundle, but we cannot delve too much into this fascinating subject). In the zeroth order approximation (modulo terms of order $\hbar$) the underlying Lagrangian submanifold describes the situation completely. But even to the first order in $\hbar$ Lagrangian geometry plays a crucial role in the analysis. For example the inner product of two semi-classical states u_1 and u_2 is computed by finding points of intersection of corresponding Lagrangian submanifolds and summing values of products $u_1\overline{u}_2$ at these points. Apparently it was due to these considerations that A. Weinstein introduced the symplectic "category", where objects are symplectic manifolds and morphisms are Lagrangian relations, and formulated his quantization program: find a "functor" from the symplectic "category" to the category of Hilbert spaces.

1.4. *Coherence and cliques*

The idea to consider Lagrangian submanifolds as morphisms has been present in the literature since the 1970s. In 1981 it was spelled out [25] and the symplectic "category" with symplectic manifolds as objects and Lagrangian submanifolds as morphisms was constructed. However it was not a true category since the composition of Lagrangian submanifolds given by symplectic reduction was not always defined. The geometry of this situation is as follows. Given three symplectic manifolds M, N and P and Lagrangian relations $\sigma \subset M_- \times N$ and $\tau \subset N_- \times P$ (here the subscript $(.)_-$ denotes the multiplication of the symplectic structure by -1) one constructs a symplectic manifold $S = M_- \times N \times N_- \times P$ and the constraint submanifold $C = M_- \times \Delta_N \times P$ of S where Δ_N is the diagonal submanifold of $N \times N_-$. The product $\rho := \sigma \times \tau$ is a Lagrangian submanifold of S. Under favourable circumstances the image of $\rho \cap C$ under the projection $\pi : S \to M_- \times P$ is also a Lagrangian submanifold, which is defined to be the composition of σ and τ. It is not hard to see that the set $\pi(\rho \cap C)$ is precisely the set-theoretic composition of relations σ and τ. However, the set $\pi(\rho \cap C)$ is guaranteed to be a smooth submanifold only if the intersections of ρ with Δ_N are transversal.

Let us look at the situation from the point of view of Linear Logic. Linear Logic is in some sense a symmetrization of the intuitionistic logic and therefore Linear Logic proofs are naturally interpreted as relations (which are symmetric in input and output) rather than functions (which are asymmetric). However the notion of a relation may seem too general;

given a relation σ between two sets X and Y it does not follow *a priori* that σ establishes any kind of functional dependence between X and Y. Probably this is not very satisfactory; the interchangeability between the input and the output certainly does not mean their independence. This motivates the idea to interpret formulas as smooth manifolds and proofs as *smooth* relations, i.e. smooth submanifolds of the ambient spaces. Indeed a smooth submanifold analytically is just a function given implicitly, by a system of equations. However this system can be solved locally and establish a true functional dependence between coordinates. Now let us look at the composition of relations. This composition amounts to elimination of variables in a system of equations: if σ and τ are given by systems

$$F(x, y) = 0 \tag{9}$$

and

$$G(y, z) = 0 \tag{10}$$

respectively than the would-be composition $\tau \circ \sigma$ is given by the conjunction of (9) and (10) with y expressed in terms of x and z. This composition fails precisely when the system above cannot be solved in y. (This is a very realistic situation. Not all algorithms terminate, therefore in reality not all morphisms are composable.) Failure to solve Eqs. (9) and (10) means that some partial derivatives are zero, and therefore the Implicit Function Theorem does not apply. If we recall that composition of morphisms is the semantical counterpart of cut-elimination, whereas cut-elimination is the process of making implicit steps of the proof explicit, then the geometric interpretation seems quite meaningful. One can say that cut-elimination theorems are logical analogues of the Implicit Function Theorem. The fact that two smooth relations do not compose is a geometric manifestation of the fact that the Implicit Function Theorem does not apply; on the logical side this would mean that the cut-elimination algorithm does not terminate.

It turns out though that the extra structure of a field of contact cones, a “coherence”, which allows us to take as morphisms between M and N only those submanifolds of $M_- \times N$ which are tangent to a certain field, is exactly what is needed in order to exclude non-composable pairs of morphisms. Recalling Girard’s slogan [11]

$$\text{FORMULAS} = \text{PLUGGING INSTRUCTIONS}$$

we may see our fields of contact cones as these instructions; they require that “plugging” of submanifolds should be transversal.

In order to apply the Implicit Function Theorem one should make sure that some partial derivatives are not zero. It is natural then to write as plugging instructions some Boolean combinations of conditions of the form $\frac{\partial F^i}{\partial x^j} = 0$. But geometrically such a Boolean combination denotes precisely a subset of the tangent bundle, which is closed under scalar multiplication. It is remarkable that this idea indeed works and leads to a sensible structure. It is remarkable also that the symplectic structure of ambient manifolds and the “Lagrangianness” of relations are necessary in order to carry out this program. As we shall see below arbitrary smooth relations are not composable in general even if one imposes plugging instructions.

1.5. Some further remarks

By no means should it be understood that the model that we discuss in this paper can be taken as some definitive final word. Rather we think of it as a first step in the direction of symplectic semantics and we would like to believe that it may serve as a building block (one among many) for future developments. We interpret only a very poor fragment of Linear Logic and the model itself is very elementary and can be modified in many ways. Apparently a more structure should be added in order to accommodate other connectives.

On the other hand there is a natural contrary question: how much of the structure is needed in order to interpret the multiplicative fragment? What is the minimal necessary degree of complexity of objects for our construction to make sense? In fact the completeness results, which we establish, can be obtained in the most general as well as in a very restrictive setting depending on what seems appropriate. We consider arbitrary symplectic manifolds precisely because we do not need anything but their local structure. Unlike the Riemannian case the local structure of a symplectic manifold is trivial in the sense that there are no local invariants; all symplectic manifolds of equal dimension are locally symplectomorphic. Therefore nothing is lost if we require all phase spaces to be vector spaces. Such a restriction on objects is perfectly consistent with "physical" intuitions behind; the phase space, which occurs most often, certainly is $\mathbf{R}^{2n}$.

A subtler point is concerned with the complexity of Lagrangian relations. Does it make sense to require further that all Lagrangian submanifolds be linear subspaces? As we shall see below our completeness results do not pass to the setting of linear subspaces. The explanation here is that it is the smooth structure (and not the algebraic one) of a vector space which matters in our model. A vector space seen as a manifold does not have any fixed algebraic structure since a manifold, which is diffeomorphic to a vector space, is diffeomorphic to it in many ways. Our interpretation is based precisely on the smooth structure and "does not see" the algebra. In physical terms: a vector space structure on the phase space depends on the arbitrary choice of coordinates and is therefore unphysical. As long as there is no fixed vector space structure it does not makes sense to speak about linear subspaces or constant fields of contact cones; the "linearity" in this case depends on the choice of coordinates. Therefore it would be quite pointless in our opinion to replace Lagrangian submanifolds with linear Lagrangian subspaces. In this case one would be forced to impose fixed vector space structures on the ambient manifolds, that is to impose a *non-trivial algebraic structure* on a manifold with a *trivial topological structure*. We think that the nonlinearity of our Lagrangian submanifolds is quite important; neither semi-classical states nor generalized solutions of PDEs are necessarily linear subspaces, and the meaning of these objects has nothing to do with a particular algebraic structure and a choice of coordinates. On the conceptual level, the analogy between Implicit Function Theorem and cut-elimination that we think to be meaningful is based on the nonlinearity of submanifolds as well.

This does not mean that the class of Lagrangian relations cannot be restricted in any way. One may for example replace smoothness with analyticity; it can make sense to consider manifolds coming from algebraic geometry, i.e. smooth symplectic varieties and smooth Lagrangian subvarieties. At the present stage though we do not see a motivation for any such restriction.

2. Symplectic geometry

Material of this section is rather standard. We refer the reader to [19] for a more detailed discussion.

2.1. Symplectic spaces and symplectic manifolds

Definition 1. A symplectic space $\langle V, \omega\rangle$ consists of a finite-dimensional vector space V and a skew-symmetric nondegenerate bilinear form ω on V.

Given a symplectic space $\langle V, \omega\rangle$ we shall always write V for $\langle V, \omega\rangle$ and given $u, v \in V$ we shall write $<u, v>$ for $\omega(u, v)$ unless it leads to confusion.

A symplectic space necessarily is even dimensional (this follows from nondegeneracy of the symplectic form).

The canonical example of a symplectic space is $\langle U \times U^*, \omega\rangle$ where U is a vector space and ω is given by

$$\omega((v_1, u_1), (v_2, u_2)) = <v_1, u_2> - <v_2, u_1>. \tag{11}$$

Like the familiar case of Euclidean spaces one may define the orthogonal "complement" with respect to the symplectic form. It follows immediately from nondegeneracy of ω that given a subspace U with $\dim U = k$ of a symplectic space V with $\dim V = 2n$, the dimension of

$$orth(U) := \{v \in V \mid <u, v> = 0\ \forall u \in U\} \tag{12}$$

is $2n - k$. A remarkable difference with the Euclidean case is that the orthogonal "complement" $orth(U)$ of a subspace U may have non-trivial intersections and even coincide with U.

Definition 2. A subspace U of $V = \langle V, \omega\rangle$ is called *isotropic* if $U \subseteq orth(U)$, *coisotropic* if $orth(U) \subseteq U$ and *Lagrangian* if it is both isotropic and coisotropic. It is called a symplectic subspace of V if it is a symplectic space under the restriction of ω to U.

The following elementary observations hopefully may give some feeling of the properties of orthogonal "complements":

Note 1. An isotropic subspace L of a symplectic space V is Lagrangian iff $\dim L = \frac{1}{2}\dim V$.

Let U, W be subspaces of a symplectic space V. Then

$$orth(orth(U)) = U,$$

$$orth(U + W) = orth(U) \cap orth(W),$$
$$orth(U \cap W) = orth(U) + orth(W).$$

If U is isotropic (coisotropic) then $orth(U)$ is coisotropic (isotropic),

if U is symplectic then $orth(U)$ is symplectic and V is symplectomorphic to $U \times orth(U)$,

if $U = V$ then $orth(U) = \{0\}$.

For (straightforward) proofs see for example [19], 1.5–1.9.

Given symplectic spaces $\langle V, \omega\rangle$, $\langle V_i, \omega_i\rangle$, $i = 1, 2$, one may construct new symplectic spaces

$$V_- := \langle V, -\omega\rangle$$

and

$$V_1 \times V_2 = \langle V_1 \times V_2, \omega_1 + \omega_2\rangle.$$

The following is clear:

Note 2. Let $\langle V, \omega\rangle$, $\langle V_i, \omega_i\rangle$, $i = 1, 2$, be symplectic spaces and L, L_1, L_2 be Lagrangian subspaces of V, V_1 and V_2 respectively. Then L and $L_1 \times L_2$ are Lagrangian subspaces of V_- and $\langle V_1 \times V_2, \omega_1 + \omega_2\rangle$ respectively.

Let us state a more technical lemma, which will be used later in this paper.

Lemma 1. *Let M_1, M_2 be symplectic vector spaces and L be a Lagrangian subspace of $M = M_1 \times M_2$. Let $\pi_i : M \to M_i$, $i = 1, 2$, be the natural projections on factors. Then*

(i) *the spaces $L_i = \pi_i(L) \subseteq M_i$ are coisotropic;*

(ii) *the space L contains $orth(L_1) \times \{0\}$ and $\{0\} \times orth(L_2)$;*

(iii) *if one of the spaces L_1, L_2 is Lagrangian then the other is Lagrangian as well and $L = L_1 \times L_2$.*

Proof. Let $v \in orth(L_1)$. Then for any vector $u = (u_1, u_2) \in L$ it holds that $<(v, 0), (u_1, u_2)> = <v, u_1> = 0$. Hence $(v, 0) \in orth(L) = L$ and $v \in \pi_1(L) = L_1$. Since v, u were arbitrary the assertions (i), (ii) are proven for L_1, and the identical argument proves (i) and (ii) for the case of L_2.

Assume now that, say, L_1 is Lagrangian. Pick any two vectors $u, u' \in L$ and let $u_i = \pi_i(u)$, $u'_i = \pi_i(u')$, $i = 1, 2$. Since L is Lagrangian we have that $0 = <u, u'> = <u_1, u'_1> + <u_2, u'_2> = <u_2, u'_2>$. Since u, u' were arbitrary it follows that $L_2 = \pi_2(L)$ is isotropic. But by (i) the space L_2 is also coisotropic, which may be the case only if L_2 is Lagrangian. Now pick arbitrary $u_1 \in L_1$, $u_2 \in L_2$. For any vector $v = (v_1, v_2)$ it holds that $<(u_1, u_2), (v_1, v_2)> = <u_1, v_1> + <u_2, v_2> = 0$. So $(u_1, u_2) \in orth(L) = L$. $\square$

In the next subsection we recall the concept of a symplectic manifold and a Lagrangian submanifold.

2.2. Symplectic manifolds

A natural generalization of the notion of a symplectic vector space is that of a *symplectic manifold*.

Definition 3. A symplectic manifold $\langle M, \omega\rangle$ consists of a smooth manifold and a closed non-degenerate 2-form ω on M.

If it does not lead to confusion we shall use the notation

$$<v_x, u_x> := \omega(x)(v_x, u_x)$$

for $v_x, u_x \in T_x M$, where $\langle M, \omega\rangle$ is a symplectic manifold.

Let $M = \langle M, \omega\rangle$ be a symplectic manifold. Since ω is nondegenerate it induces an isomorphism $TM \cong T^*M$ and via this isomorphism a 2-form is defined on cotangent vectors as well. It follows that ω induces a Lie bracket on the space of smooth functions on M (the Poisson bracket) by

$$\{f, g\}(x) = <df(x), dg(x)>.$$

A remarkable fact is that all symplectic manifolds of the same dimension locally look alike. We state without a proof the following classical theorem.

Theorem 1 (*Darboux Theorem*). *Let $M = \langle M, \omega\rangle$ be a symplectic manifold.*

Then for any point $x \in M$ there exists a local coordinate system of the form (1) *such that relations* (2) *hold, and the symplectic form reads*

$$\omega = \sum dp_i \wedge dq^i. \tag{13}$$

The tangent space to a symplectic manifold is a symplectic vector space. Further, a submanifold C of a symplectic manifold M is isotropic, coisotropic, Lagrangian or symplectic if for any $x \in C$ the tangent space T_xC is respectively an isotropic, a coisotropic, a Lagrangian or a symplectic subspace of T_xM. Our main interest is in Lagrangian submanifolds. They share most of the properties of Lagrangian subspaces of symplectic vector spaces. In particular it is clear that Note 2 lifts to the setting of manifolds and the class of Lagrangian submanifolds is closed under Cartesian products and is invariant under multiplication of the symplectic structure by -1. One can also deduce from Note 1 that a Lagrangian submanifold is necessarily of dimension equal to half of that of the ambient symplectic manifold.

A typical example of a Lagrangian submanifold of a symplectic manifold M is as follows. Pick a point x, pick a canonical coordinate system (q, p) near x. Let L be defined by the equations

$$q^i(x) = 0, \; i = 1, \ldots, n.$$

Then in a neighborhood U of x the tangent bundle TU is trivial, $TU \cong U \times T_xM$ and the tangent bundle TL in this trivialization is given by

$$TL = \{v \in T_LU | <v, dq^i> = 0, \; i = 1, \ldots, n\}. \tag{14}$$

(Here T_LU is the pull-back of the tangent bundle TU by the immersion $i : L \to U$, i.e. T_LU is the set of those vectors, which are tangent to U at points of L.)

It follows from relations (2) and the definition of the Poisson bracket that for each $y \in M$ the subspace l_y of T^*_yM spanned by $dq^1(y), \ldots, dq^n(y)$ is isotropic and counting dimensions we see that l_y is Lagrangian. On the other hand by (14) for each $y \in L$ the space T_yL lies in the annihilator of l_y and counting dimensions again we see that actually $T_yL = Ann(l_y)$. Hence identifying by means of the symplectic form the spaces T_yM and T^*_yM we get that $T_yL = orth(l_y) = l_y$. Hence T_yL is Lagrangian.

From this reasoning a (rather banal) note follows, which will be used in the sequel.

Note 3. Let M be a symplectic manifold. Let $x_1, \ldots, x_k \in M$. There exists a Lagrangian submanifold L of M such that $x_1, \ldots, x_k \in L$.

Proof. By the reasoning above for each $i = 1, \dots, m$ there exists a Lagrangian submanifold L_i of M containing x_i. Taking as L the disjoint union $L_1 \cup \dots \cup L_m$ we obtain a desired manifold. □

In the note above we did not require L to be connected as this does not seem relevant for our purposes. It seems plausible that (at least under some fairly general assumptions) one can make it connected after a little extra work. Let us observe however that a submanifold in the statement of the note cannot be replaced with a subspace. In fact if M is a symplectic vector space and $x_1, \dots, x_m$ is a basis of M then the only subspace, which contains all these vectors, is M itself. In particular no Lagrangian subspace of M will work.

The Darboux theorem says that from the point of view of local phenomena there is, essentially, only one symplectic manifold of the given dimension. It is useful to think of this manifold as of the *cotangent bundle*. In the next subsection we describe the canonical symplectic structure on the cotangent bundle.

2.3. Symplectic structure on the cotangent bundle

Let Q be a manifold and T^*Q be its cotangent bundle.

At any point $(q, u_q) \in T^*Q$ the tangent space $T_{(q,u_q)}T^*Q$ to T^*Q is isomorphic to $T_qQ \times (T_qQ)^*$.

The space T_qT^*Q is equipped with an invariant $1-$form α (the Liouville form) given locally by

$$\alpha(u, v) = \langle u, v\rangle, \tag{15}$$

where $v \in T_qQ$, $u \in (T_qQ)^*$ and with a $2-$form $d\alpha$ (the canonical symplectic form) given locally by

$$d\alpha((u_1, v_1), (u_2, v_2)) = \langle u_1, v_2\rangle - \langle u_2, v_1\rangle \tag{16}$$

(compare with (11)). In fact we have these forms invariantly defined on the whole T^*Q.

It may be shown that the form $d\alpha$ is non-degenerate. Thus the cotangent bundle T^*Q is canonically a symplectic manifold. The physical meaning is that the base Q is the configuration space of the system and T^*Q is the space of all possible kinematic states.

Canonical coordinates of the Darboux theorem have very simple meaning in this case.

If U is a sufficiently small neighborhood in Q and $q = (q^1, \dots, q^n)$ are local coordinates on U then T^*U is diffeomorphic to $U \times \mathbf{R}^n$, where $n = \dim Q$, and any covector $u \in T^*U$ has unique representation $u = (q^1, \dots, q^n, p_1dq^1 + \dots + p_ndq^n)$. The coordinates $(p, q) = (p_1, \dots, p_n, q^1, \dots, q^n)$ are called *canonical coordinates* on the cotangent bundle.

Let us mention several natural examples of Lagrangian submanifolds of T^*Q. These are:

- each fiber of the projection $T^*Q \to Q$;
- the zero-section of T^*Q, which is usually identified with the base Q;
- for any smooth function $u \in C^\infty(Q)$ the graph of its differential du, i.e. the set $\{(du(q), q) | \, q \in Q\}$ is a Lagrangian submanifold of T^*Q;

- more generally, we may replace du in the previous example with any closed 1-form on Q.

In the next section we recall the fundamental concept of symplectic reduction.

2.4. Symplectic reduction

We proceed to the operation of *symplectic reduction*, which is fundamental in symplectic geometry.

Let M, σ be manifolds and $i : \sigma \to M$ be a smooth map.

Recall that (σ, i) (or just σ for short) is an *immersed submanifold* of M or an *immersion* in M if for any point $x \in \sigma$ there exists a neighborhood U of x such that i takes U diffeomorphically onto a submanifold of M. If the manifold M is symplectic and for each $x \in \sigma$ there exists a neighborhood U of x such that $i(U)$ is a Lagrangian submanifold of M then the immersion (σ, i) is called *Lagrangian*.

In this paper we shall understand the notation $\sigma \subset M$ as "σ is immersed in M".

Two immersions (σ_1, i_1) and (σ_2, i_2) are equal if there exists a diffeomorphism $f : \sigma_1 \to \sigma_2$ such that $i_1 = f^* i_2$. Two immersions (σ_1, i_1) and (σ_2, i_2) in a manifold M are said to be *transversal* if for any two points $x_i \in \sigma_i$, $i = 1, 2$, such that $i_1(x_1) = i_2(x_2) = x$, it holds that the tangent spaces to the images $i_1(\sigma_1)$ and $i_2(\sigma_2)$ span the whole $T_x M$. In this case one may form their *fiber product* $(\sigma_1 \cap \sigma_2, i_1 \times_M i_2)$ (or $\sigma_1 \cap \sigma_2$ for short) where

$$\sigma_1 \cap \sigma_2 = \{(x_1, x_2) |\, x_i \in \sigma_i,\ i = 1, 2,\ i_1(x_1) = i_2(x_2)\},$$

and

$$i_1 \times_M i_2(x_1, x_2) = i_1(x_1).$$

Given a manifold M, a subset D of TM is called a *distribution* (or a subbundle) if for any $x \in M$ the set $D \cap T_x M$ is a linear space and $\dim(D \cap T_x M) = k$ for some fixed k independent of x. (One uses often a more general definition of a distribution where the second condition is dropped.) A distribution D is said to be *smooth* if the assignment $x \mapsto D \cap T_x M$ is smooth. More precisely D is smooth if for any vector $v = v_x \in D$ tangent to M at the point x there exists a smooth vector field ξ on a neighborhood of x such that $\xi \subset D$ and $\xi(x) = v_x$.

A submanifold σ of M is said to be an *integral submanifold* of D if $T\sigma \subseteq D$. A distribution D is said to be *completely integrable* if for any $x \in M$ there is an integral submanifold σ of D passing through x.

It is well known that integrable distributions of dimension greater than 1 are by no means a common thing. (One-dimensional distributions are simply vector fields.) The classical Frobenius theorem states:

Theorem 2 (*Frobenius Theorem*). *Let M be a manifold and D be a smooth distribution on M. Then D is completely integrable iff for any two vector fields ξ, η lying in D their commutator $[\xi, \eta]$ lies in D as well (such distributions are called involutive).*

For a proof see any textbook on differential geometry, for example [18].

Given a completely integrable smooth distribution D on a manifold M, a connected integral submanifold σ of D is called *maximal* if σ is not contained in any connected integral submanifold of D other than itself. The partition of M into maximal integral submanifolds is called the *foliation* of M generated by D and maximal integral submanifolds of D are called *leaves* of this foliation (leaves of D for short). Finally the foliation generated by D is *simple* if there exists a smooth manifold structure on the set M/D of leaves of D such that the natural projection $\pi : M \to M/D$ is smooth.

Now let M be a symplectic manifold and C a coisotropic submanifold of M. Assume also that for each $x \in X$ the dimension k of $orth(T_x C)$ is independent of x. Since for each $x \in C$ the space $T_x C$ is coisotropic, i.e. $orth(T_x C) \subseteq T_x C$, we have that the distribution

$$orth(TC) := \bigcup_{x \in C} orth(T_x C)$$

(called the *null distribution*) lies in TC.

Let us denote $orth(TC)$ by $C^{\perp}$. Suppose that $C^{\perp}$ is a smooth distribution on C. The following theorem holds:

Lemma 2. *With notations as above the distribution $C^{\perp}$ is integrable. Suppose that the foliation generated by $C^{\perp}$ is simple. Let $\overline{C}$ be the set of leaves of $C^{\perp}$ and $\pi : C \to \overline{C}$ be the natural projection. Then the manifold $\overline{C}$ has a symplectic structure given by a well defined symplectic form $\overline{\omega}$:*

$$\overline{\omega}(\pi(x))(T\pi(v_x), T\pi(u_x)) = \omega(x)(v_x, u_x)$$

where $x \in C$, $u_x, v_x \in T_x C$.

Furthermore if L is an immersed Lagrangian submanifold of M transversal to C then $\pi(L \cap C)$ is an immersed Lagrangian submanifold of $\overline{C}$ and π restricted to $L \cap C$ is an immersion.

For a proof see for example [19], 3.14.3 and 3.14.19.

With notations as above we shall call the projection $C \to \overline{C}$ *symplectic reduction* and the manifold $\overline{C}$ the *reduced space*. The foliation of C into leaves of $C^{\perp}$ is the *characteristic foliation*. Physically the coisotropic submanifold C is a *constraint manifold*; if C is the level set of functions $f_1, \ldots, f_k$ then $f_1, \ldots, f_k$ are *constraints*. The reduced space is the phase space of a constrained system; extra degrees of freedom are eliminated from equations of motion by means of constraints.

The Hamilton equations (3) have a beautiful geometric interpretation in the context of symplectic reduction. Given a symplectic manifold M and a Hamiltonian $H \in C^{\infty}(M)$ assume that $a \in \mathbf{R}$ is a noncritical value of H, i.e. that the set $C = H^{-1}(a)$ is a submanifold of M. Then the manifold C is coisotropic and its null-distribution $C^{\perp}$ is one dimensional and is spanned by the Hamiltonian vector field of H. The characteristic foliation of C is the foliation into the integral curves of the Hamilton equations (3).

This interpretation can be reversed. The operation of symplectic reduction amounts from the analytic point of view to integration of a system of PDE. In practice this is done by successive integration of Hamilton equations with constraint functions playing the role of Hamiltonians. Thus, ultimately, symplectic reduction is a *dynamical* procedure: the reduced space is the space of evolutions determined by constraint Hamiltonians.

In the next section we describe the symplectic "category" as defined by A. Weinstein in [25] and then introduce the category of coherent phase spaces.

3. Coherent phase spaces

3.1. The symplectic "category"

In the symplectic "category" one takes symplectic manifolds as objects and Lagrangian submanifolds of $A_- \times B$ (so-called *canonical relations*) as morphisms between symplectic manifolds A and B. The idea to consider Lagrangian submanifolds as morphisms originated apparently in the context of asymptotic quantization. We refer the reader to [3] for a brief introduction into this circle of methods and ideas, with the emphasis on the symplectic "category".

One may wonder what is the natural notion of a morphism between symplectic manifolds. If we attempt to define a morphism between symplectic manifolds in the obvious way as a smooth map which preserves symplectic structure then we immediately find that such morphisms are very scarce. Basically the only smooth maps which may preserve symplectic structure are immersions. On the other hand there are plenty of canonical relations between two symplectic manifolds. An instructive example is as follows.

Let Q and P be two general (not symplectic) manifolds and $f : Q \to P$ be a smooth map. Recall that the cotangent bundles T^*Q and T^*P are symplectic manifolds. Now, let $F \subseteq Q \times P$ be the graph of f and

$$TF^0 = \{\phi \in T^*(Q \times P) \cong T^*Q \times T^*P \mid <\phi, v> = 0 \text{ for all } v \in TF\}$$

be the annihilator of TF. Then, as follows from the definition of the canonical symplectic structure on a cotangent bundle, TF^0 is a Lagrangian submanifold of $T^*Q \times T^*P$ and after the transformation

$$\sigma : T^*Q \times T^*P \to (T^*Q)_- \times T^*P,\ \sigma : (u, v) \mapsto (-u, v)$$

the image $\sigma(TF^0)$ becomes a canonical relation on $T^*Q \times T^*P$. Thus a smooth map f between Q and P lifts to a canonical relation between corresponding cotangent bundles called the *cotangent lift* of f. Explicitly in local coordinates this lift is given for $f(q) = (f^1(q), \ldots, f^m(q))$ (m is the dimension of the target manifold P) by

$$T^*f = \left\{\left(\sum \frac{\partial f^i}{\partial q^1}\phi_i, \ldots, \sum \frac{\partial f^i}{\partial q^n}\phi_i, q^1, \ldots, q^n,\right.\right.$$
$$\left.\left.\phi_1, \ldots, \phi_m, f^1(q), \ldots, f^m(q)\right)\right\}.$$

One may check that this lift is functorial, i.e. the composition of maps lifts to the composition of relations. Note that in general a smooth map does not lift to any map between cotangent bundles. One may say that there exists a functor from the category of smooth manifolds to the symplectic "category".

Now let us discuss how to compose canonical relations. Given two canonical relations $\sigma \subset M_- \times N$ and $\tau \subset N_- \times P$ one attempts to define their composition set-theoretically i.e.

$$\tau \circ \sigma := \{(x, z) \in M \times P | \exists y \in N \text{ s.t. } (x, y) \in \sigma \text{ and } (y, z) \in \tau\}. \tag{17}$$

The problem with this composition is that $\sigma \circ \tau$ is not in general a submanifold of $M \times P$. However if $\sigma \times \tau$, a Lagrangian submanifold of $M_- \times N \times N_- \times P$, is transversal to the coisotropic manifold

$$C = \{(x, y_1, y_2, z) \in M_- \times N \times N_- \times P | y_1 = y_2\} \tag{18}$$

then Lemma 2 applies and $\sigma \circ \tau$ is easily seen to be a canonical relation on $M_- \times P$.

Indeed the null distribution $C^\perp$ consists of vectors of the form $(0, v, v, 0) \in TM \times TN \times TN \times TP$ and leaves of $C^\perp$ are nothing but manifolds of the form $\{x\} \times \Delta_N \times \{z\}$ where $x \in M$, $z \in P$ and $\Delta_N = \{(y, y)| y \in N\}$ is the diagonal submanifold of $N \times N$. Hence the reduced space $\overline{C}$ is just $M_- \times P$ and symplectic reduction $\pi : C \to \overline{C}$ is the natural projection on the first and the fourth factor. Since $\sigma \circ \tau$ was defined exactly as the image of $\sigma \times \tau \cap C$ under π we conclude that $\sigma \circ \tau$ is indeed a Lagrangian submanifold of $M_- \times P$.

There exists a special class of canonical relations which are composable with anything provided that domain and codomain match.

Note 4. Let M, N be symplectic manifolds, $f : M \to N$ a symplectomorphism. Then the graph F of f is a canonical relation on $M_- \times N$, and for any pair of symplectic manifolds M', N' and canonical relations $\sigma \subset M'_- \times M$, $\tau \subset N_- \times N'$ the compositions $F \circ \sigma$, $\tau \circ F$ are defined.

Proof. Pick a point $y = (x, f(x))$ of F, and let $w = (u, v) \in T_y(M_- \times N)$, where $u \in T_x(M_-)$, $v \in T_{f(x)}N$. Assume $w \in orth(T_y F)$. Then for any $\alpha \in T_x(M_-)$ it holds that $<u, \alpha> - <v, T_x f \alpha> = 0$. For a fixed u this relation becomes a system of equations on v. The map $T_x f$ preserves the symplectic form hence $v = T_x f(u)$ is a solution. By nondegeneracy of the symplectic form this solution is unique. Thus $w = (u, T_x f(u)) \in T_y F$. On the other hand if $w = (u, T_x f(u)) \in T_y F$ then again w annihilates all vectors in $T_y F$.

So $orth(T_y F) = T_y F$ and $T_y F$ is Lagrangian.

Finally let M' be another symplectic manifold and $\sigma \subset M \times M'$ be a canonical relation. Then as is easy to see the set-theoretic composition $F \circ \sigma$ is nothing but the image of σ under the symplectomorphism $id \times f$. Since symplectomorphisms take Lagrangian submanifolds to Lagrangian submanifolds it follows that $F \circ \sigma$ is a canonical relation.

The case of N' and τ is analogous. $\square$

But in general composition of canonical relations is not well defined and the symplectic "category" is not a true category.

Nevertheless the symplectic "category" is quite an interesting mathematical object. From now on we shall denote this object by **Wei**. Let us explore the structure of **Wei**.

(i) Whenever the composition of canonical relations is defined it is associative. This follows from the associativity of the set-theoretic composition of ordinary relations.

(ii) For any symplectic manifold M there exists a (unique) canonical relation $id_M \subset M_- \times M$ such that for any other symplectic manifold N and canonical relations $\sigma \subset M_- \times N$, $\tau \subset N_- \times M$ the compositions $\sigma \circ id_M$, $id_M \circ \tau$ are defined and

$$\sigma \circ id_M = \sigma, \; id_M \circ \tau = \tau. \tag{19}$$

Of course one takes as id_M the diagonal submanifold $\Delta = \{(x, x)|\, x \in M\}$. This is the graph of a symplectomorphism (namely of the identity map) hence by Note 4 it is Lagrangian and is composable with everything. Relations (19) obviously hold.

(iii) There exists a symmetric "tensor product" in **Wei**, namely the usual Cartesian product of symplectic manifolds. This operation is symmetric in the sense that there exists a (natural) family of canonical isomorphisms $c_{MN} : M \times N \to N \times M$ indexed by pairs of objects of **Wei**. Each symmetry isomorphism c_{MN} is nothing but the graph of the symplectomorphism $M \times N \to N \times M$ induced by the permutation of factors.

Physically: putting two classical systems together corresponds to the Cartesian product of phase spaces; after quantization this Cartesian product becomes the tensor product of Hilbert spaces.

(iv) There exists a "unit object" neutral with respect to the tensor product, namely the single point manifold I with the trivial symplectic structure.

(v) There exists a contravariant involution $(.)^\perp$ which takes a symplectic manifold M to M_- and a canonical relation $\sigma \subset M_- \times N$ to the adjoint relation $\sigma^\perp \subset N_- \times M$ obtained by switching N and M.

(vi) There exists a bijection between the hom-sets

$$\mathbf{Wei}(M \otimes N, P^\perp) \cong \mathbf{Wei}(M, (N \otimes P)^\perp),$$

which means that **Wei** is a $*$-autonomous "category" (a precise definition to be given later).

(vii) There is a natural family of isomorphisms

$$(M \otimes N)^\perp \cong M^\perp \otimes N^\perp,$$

which means that **Wei** is a compact-closed "category" (definition again later).

These properties correspond to known properties of finite-dimensional Hilbert spaces. This observation lies in the basis of the quantization program of A. Weinstein: to find a functor from the symplectic "category" to the category of Hilbert spaces, which respects the correspondence above [3]. (Although this program can be considered only as heuristic, the symplectic "category" is not a true category, on the other hand infinite-dimensional Hilbert spaces do not enjoy the properties listed above.)

In the next subsection we turn the symplectic "category" **Wei** into a true category.

3.2. *Coherent phase spaces*

In this section we define the category **CohPS** of coherent phase spaces.

Let V be a vector space.

Definition 4. A nonempty subset A of V is a contact cone if for any $v \in A$ the whole line $\{tv|\, t \in \mathbf{R}\}$ lies in A.

It is easy to see that adding the zero vector to the complement of a contact cone we obtain a contact cone again. Thus each contact cone determines a partition of $V - \{0\}$ (or of the projectivization $\mathbf{P}V$ if the reader prefers) into two disjoint subsets.

To keep with the syntax of linear logic we shall denote the complement of a contact cone A by $A^{\perp}$.

Further it is immediate that the following operations are well defined:

Definition 5. Let $\langle V_i, \omega_i\rangle$, $i = 1, 2$ be symplectic spaces, $A_i \subset V_i$ — contact cones. Let $V = \langle V_1 \times V_2, \omega_1 + \omega_2\rangle$.

The tensor and cotensor products of A_1 and A_2 are given by

$$A_1 \otimes A_2 := \{(v_1, v_2)|\ v_i \in A_i,\ i = 1, 2\},$$
$$A_1 \wp A_2 = (A_1^{\perp} \otimes A_2^{\perp})^{\perp} = \{(v_1, v_2)|\ 0 \neq v_1 \in A_1 \text{ or } 0 \neq v_2 \in A_2\} \cup \{0\}.$$

The linear implication is given by

$$A_1 \multimap A_2 = A_1^{\perp} \multimap A_2 = \{(v_1, v_2)|\ v_1 \in A_1 \text{ implies } 0 \neq v_2 \in A_2\}.$$

Now we lift our definitions to the setting of manifolds.

Definition 6. Let M be a symplectic manifold. A subset A of the tangent bundle TM of M is a field of contact cones if for any $x \in M$ the set $A(x) := T_x M \cap A$ is a contact cone in $T_x M$.

Obviously a field A of contact cones on a symplectic manifold M determines a partition of the tangent bundle TM of M into two complementary subsets — just like the case of vector spaces (more precisely, into two subsets whose intersection is the zero section).

Definition 7. A coherent phase space is a pair $\langle M, A\rangle$ where M is a symplectic manifold and A is a field of contact cones on M.

Usually our notation for a coherent phase space will be $\langle M_A, A\rangle$ or $\langle A, \tilde{A}\rangle$. Also, if it does not lead to confusion, we will denote a coherent phase space and the underlying symplectic manifold by the same letter.

Definition 8. A state of a coherent phase space $A = \langle M, A\rangle$ is an immersed Lagrangian submanifold of M tangent to A at every point.

This terminology is motivated by our interpretation of Lagrangian submanifolds as semiclassical states.

Now we lift the operations defined on contact cones to coherent phase spaces.

Definition 9. Given two coherent phase spaces $\langle M_1, A_1\rangle$ and $\langle M_2, A_2\rangle$ their tensor product $A_1 \otimes A_2$ and cotensor product $A_1 \wp A_2$ as well as negation $A_i^{\perp}$ and linear implication $A_1 \multimap A_2$ are given by pointwise operations on corresponding contact cones.

More precisely tensor and cotensor products of A_1 and A_2 are fields of contact cones on $M_1 \times M_2$ given by:

$$\begin{aligned} A_1 \otimes A_2 &= \{v \in T_{x_1}M_1 \times T_{x_2}M_2|\ v \in A_1(x_1) \otimes A_2(x_2),\ x_i \in M_i,\ i = 1, 2\}, \\ A_1 \wp A_2 &= \{v \in T_{x_1}M_1 \times T_{x_2}M_2|\ v \in A_1(x_1) \wp A_2(x_2),\ x_i \in M_i,\ i = 1, 2\}, \end{aligned}$$

where $T(M_1 \times M_2)$ is identified with $TM_1 \times TM_2$.

Negation of A_1 is the field of contact cones on $(M_1)_-$ given by

$$A_1^\perp = \{v \in T_x(M_1) |\ v \in A_1(x)^\perp\}$$

and linear implication $A_1 \multimap A_2$ is the field

$$A_1 \multimap A_2 = \{v \in T_{x_1}M_1 \times T_{x_2}M_2 |\ x \in A_1(x_1) \multimap A_2(x_2),\ x_i \in M_i,\ i = 1, 2\}.$$

Note that if we agree that a tangent vector is really a pair of points which are "infinitely close to each other" then our definition of tensor and cotensor translates literally to the definition of corresponding operations in ordinary coherent spaces. A field of contact cones itself being a subset of the tangent bundle may be seen then as a set of infinitely close pairs, i.e. as an infinitesimal relation. This suggests that our construction is indeed a smooth version of coherent spaces of Girard.

Now we are ready to define our category.

Definition 10. The category **CohPS** of coherent phase spaces consists of:

the class of coherent phase spaces as objects,

for each pair of coherent phase spaces A and B the set of states of $A^\perp \wp B = A \multimap B$ as morphisms between A and B.

For two morphisms $\sigma : \langle M_X, X\rangle \to \langle M_Y, Y\rangle$ and $\tau : \langle M_Y, Y\rangle \to \langle M_Z, Z\rangle$ their composition is defined by the formula (17).

Theorem 3. *The definition above is consistent;* **CohPS** *is a category.*

Proof. Since the composition in **CohPS** is given by the same formula as in **Wei** it follows that this composition is associative whenever defined. We only have to check that the composition in **CohPS** is always defined.

Lemma 3. *Let $\langle M_X, X\rangle$, $\langle M_Y, Y\rangle$ and $\langle M_Z, Z\rangle$ be coherent phase spaces. Let σ and τ be states of $X \multimap Y$ and of $Y \multimap Z$ respectively.*

Then the set $\tau \circ \sigma$ defined by (17) is a state of $X \multimap Z$.

Proof. We will assume for simplicity that M_Y is connected. If this is not the case one has to repeat the argument below for each connected component M_i of M_Y such that $\sigma \times \tau$ meets $M_X \times M_i \times M_i \times M_Z$.

Consider the manifold $M = (M_X)_- \times M_Y \times (M_Y)_- \times M_Z$ with the natural symplectic structure of a Cartesian product. The manifold $\sigma = \sigma_1 \times \sigma_2 \subset M$ is a Lagrangian immersion in M.

Construct the *constraint submanifold* $C := \{(x, y_1, y_2, z) \in M|\ y_1 = y_2\}$.

The manifold C is coisotropic. Indeed for any point $c = (x, y, y, z) \in C$ the tangent space T_cC is given by

$$T_cC = \{(u, v_1, v_2, w) \in T_cM \cong T_xM_X \times T_yM_Y \times T_yM_Y \times T_zM_Z |\ v_1 = v_2\} \quad (20)$$

and by a straightforward calculation one sees that

$$orth(T_cC) = \{(0, v, v, 0) \in T_cC |\ v \in T_yM_Y\}. \quad (21)$$

Thus the null distribution $C^\perp$ is nothing else but

$$C^\perp = \bigcup_{x\in M_X, z\in M_Z} T(\{x\}\times \Delta_Y \times \{z\}) \subseteq TC.$$

Hence we deduce:

Note 5. Each submanifold of M of the form $\{x\}\times\Delta_Y\times\{z\}$ where $x\in M_X$, $z\in M_Z$ is a leaf of the foliation generated by $C^\perp$. Conversely each leaf of $C^\perp$ is of this form.

Proof. One direction is obvious.

Conversely let $\gamma\subset C$ be a leaf of $C^\perp$. Let $a_i=(x_i,y_i,y_i,z_i)$, $i=0,1$, be two distinct points of γ. Let $\phi:[0,1]\to\gamma$ be a path joining a_0 and a_1. Since for any $t\in[0,1]$ it holds that $\phi'(t)\in T_{\phi(t)}\gamma\subseteq C^\perp$ we have that $\phi'(t)$ is of the form $\phi'(t)=(0,v_{\phi(t)},v_{\phi(t)},0)$, where $v_{\phi(t)}\in T_{\phi(t)}M_Y$. So denoting $x(t)=\pi_1(\phi(t))$, $z(t)=\pi_4(\phi(t))$ where π_1,π_4 are the projections from M on the first and on the fourth coordinate respectively we have that $x(t)$ and $z(t)$ are constant along ϕ hence $x_0=x_1=x$ and $z_0=z_1=z$. Thus we see that $\gamma\subseteq\{x\}\times\Delta_Y\times\{z\}$. But then it follows from the definition of a maximal integral submanifold of $C^\perp$ and the first claim that $\gamma=\{x\}\times\Delta_Y\times\{z\}$. $\square$

It follows that the set of leaves of $C^\perp$ equals $M_X\times M_Z$ and this foliation is simple,

$$\overline{C}=C/C^\perp\cong M_X\times M_Z. \tag{22}$$

So by Lemma 2 the manifold $\overline{C}$ has a natural symplectic structure induced by the natural projection and a straightforward computation shows that the diffeomorphism (22) is actually a symplectomorphism.

It is easy to see that the spaces $C^\perp\cap T_xC$ and $T_x\sigma$ are of intersection zero for each point $x\in C\cap\sigma$.

Indeed, let $u\in T_x\sigma\cap C^\perp$. It follows from (21) that u is of the form $u=(0,v,v,0)$ for some $v\in TM_Y$. Now since $\sigma=\sigma_1\times\sigma_2$ we have that $(0,v)\in T\sigma\subset X\multimap Y=X^\perp\wp Y$ and $(v,0)\in T\sigma_2\subset Y\multimap Z=Y^\perp\wp Z$. By the definition of the cotensor product it holds then that $v\in Y$, and $v\in Y^\perp$, so $v=0$, and $u=(0,0,0,0)$.

It follows from an easy lemma below that C and σ are transversal.

Lemma 4. *Let V be a symplectic vector space. Let $W\subseteq V$ be a coisotropic subspace and $L\subseteq V$ be a Lagrangian one. Assume that $L\cap orth(W)=\{0\}$. Then W and L are transversal.*

Proof. By Note 1

$$orth(L+W)=orth(L)\cap orth(W)=L\cap orth(W)=\{0\}$$

hence $L+W=orth(\{0\})=V$. $\square$

Note that it was the "Lagrangianness" of σ_1 and σ_2 which was crucial for establishing this transversality and, consequently, for carrying out the argument. In particular two general smooth relations may not compose even in the presence of "plugging instructions", i.e. of fields of contact cones (an interested reader will easily find an example).

Thus by Lemma 2 we have that $\overline{\sigma}:=\pi(\sigma_1\times\sigma_2)$ is a Lagrangian immersion in $\overline{C}\cong M_X\times M_Z$.

Let us check that $\overline{\sigma}$ is a state in $X \multimap Z$.

Assume that a vector $v = (v_1, v_2) \in T\overline{\sigma}$, $v_1 \in TM_X$, $v_2 \in TM_Y$, is such that $v \in X \otimes Z^\perp$. Then $v_1 \in X$, $v_2 \in Z^\perp$.

The tangent bundle $T\overline{\sigma}$ is the image of $T\sigma$ under the map $T\pi : T\sigma \to T\overline{\sigma}$, tangent to the symplectic reduction π. It follows that there exists some $u \in T\sigma$ such that $T\pi(u) = v$. That means that $u = (v_1, \tilde{u}, \tilde{u}, v_2)$ for some $\tilde{u} \in TM_Y$.

Since $(v_1, \tilde{u}) \in T\sigma_1$, and σ_1 is a state of $X \multimap Z$ the relation $v_1 \in X$ implies $0 \neq \tilde{u} \in Y$ unless $v_1 = 0$, $\tilde{u} = 0$.

By the same reasoning, since σ_2 is a state of $Y \multimap Z \cong Z^\perp \multimap Y^\perp$ the relation $v_2 \in Z^\perp$ implies $0 \neq \tilde{u} \in Y^\perp$ or $v_2 = 0$, $\tilde{u} = 0$.

It follows that $v_1 = 0$, $\tilde{u} = 0$, and $v_2 = 0$, hence $v = 0$. $\square$

It remains to show that for any coherent phase space $\langle M_A, A\rangle$ the diagonal submanifold $\Delta \subset M_A \times M_A$ is a morphism in **CohPS**. This is straightforward. $\square$

We have seen in Section 3.1 that there exists a "functor" $T^*(.)$, from the category **Man** of manifolds and smooth maps to the symplectic "category" which sends a manifold to its cotangent bundle and a smooth map to its cotangent lift. Remarkably this "functor" lifts to a true functor into the category of coherent phase spaces.

Definition 11. Given a manifold Q let us define the coherent cotangent bundle of Q, which we denote by abuse of notation by T^*Q, as follows. It is a coherent phase space T^*Q, which as a manifold is indeed the cotangent bundle of Q and whose field of contact cones consists of vertical vectors. That is

$$T^*Q = \langle T^*Q, \{v \in TT^*Q |\ v \text{ is tangent to a fiber of } T^*Q\}\rangle.$$

After this definition of a cotangent bundle it is easy to prove the following:

Note 6. The cotangent lift functor $T^*(.)$ described above is indeed a functor **Man** $\to$ **CohPS**.

4. Linear Logic

In this section we recall the syntax of (Multiplicative) Linear Logic and the machinery of proof-nets.

4.1. The syntax of Linear Logic

Recall that formulas of **MLL** are built from literals $p_0, p_0^\perp, p_1, p_1^\perp, \ldots, p_n, p_n^\perp, \ldots$ by means of binary connectives $\otimes$ (times) and $\wp$ (par). Linear negation $A^\perp$ of a formula A is defined inductively:

$$(p^\perp)^\perp := p,$$

$$(A \otimes B)^\perp := A^\perp \wp B^\perp,\ (A \wp B)^\perp := A^\perp \otimes B^\perp.$$

Linear implication is defined by

$$A \multimap B = A^\perp \wp B.$$

An **MLL**-*sequent* is an expression of the form $\vdash A_1, \ldots, A_n$, where A_i, $i = 1, \ldots, n$, are **MLL** formulas.

Definition 12. Multiplicative Linear Logic (**MLL**) contains the following rules:

$$\vdash A, A^{\perp} \text{ (Identity)},$$

$$\frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta} \text{ (Cut)},$$

$$\frac{\vdash A_1, \ldots, A_n}{\vdash A_{\rho(1)}, \ldots, A_{\rho(n)}}, \rho \in S_n \text{ (Exchange)},$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \text{ (Par)} \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \text{ (Times)}.$$

Definition 13. The system **MLL + Mix** is obtained from **MLL** by adding the rule

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \text{ (Mix)}.$$

The necessity to consider the Mix rule came from semantical considerations. Most known models of **MLL** are actually models of **MLL+Mix**.

It goes without saying that both **MLL** and **MLL+Mix** enjoy cut-elimination.

One of the most convenient tools for the study of Multiplicative Linear Logic is provided by the formalism of proof-nets. We will use this formalism in our proofs of completeness theorems for interpretation of **MLL** in **CohPS**.

A proof-net is a graph whose vertices are labeled by **MLL** formulas, which satisfies certain properties. A precise definition is to be given in the next subsection.

4.2. *Proof-nets*

A *proof-structure* is a graph whose vertices are labeled by occurrences of **MLL** formulas and whose edges are built via links of the following forms:

$$\frac{id}{A \quad A^{\perp}} \qquad \frac{A \quad A^{\perp}}{\text{cut}}$$

$$\frac{A \quad B}{A \otimes B} \qquad \frac{A \quad B}{A \wp B}$$

(the Identity link, the Cut link, the Times and the Par links respectively).

It is clear how to associate a proof-structure with an **MLL**-proof. We interpret Identity axioms as identity links. In order to interpret a proof π obtained from proofs π_1 and π_2 by means of, say, the Times rule, while π_1 and π_2 are interpreted by proof-structures ρ_1 and ρ_2 respectively, we draw a Times link between appropriate vertices of ρ_1 and ρ_2, etc. Thus, there is a simple translation from proofs to proof-structures. Furthermore a Cut-elimination algorithm for proof-structures also exists and is parallel to the Cut-elimination for proofs.

The class of proof-nets consists exactly of those proof-structures which come from proofs.

There are several equivalent criteria for a proof-structure to be a proof-net. The most frequently used in modern literature (though not the original one due to Girard [10]) is due to Danos and Regnier [8].

A *switching* α of a proof structure ρ is an assignment of a *choice* $S(L) \in \{\text{right}, \text{left}\}$ to each Par-link L of ρ or, equivalently, a graph obtained from ρ by deleting for each Par-link L one of the two edges of ρ which form L.

Definition 14. A proof-structure ρ is an **MLL + Mix** *proof-net* if for every switching α of ρ the graph α is acyclic. A proof-structure ρ is an **MLL** *proof-net* if for every switching α of ρ the graph α is acyclic and connected.

The following theorem sheds light on the definition above.

Theorem 4 (*[8]*). *If a proof-structure is an* **MLL** *(***MLL + Mix***) proof-net then it comes from an* **MLL** *(***MLL + Mix***) proof.*

The correspondence {proof $\mapsto$ proof-net} is bijective modulo inessential permutation of rules. This interpretation also commutes with Cut-elimination. Thus a proof-net may be thought of as a "canonical" representative of a class of proofs having the same structure.

Below we will be concerned only with **MLL + Mix** proof-nets and therefore the prefix **MLL + Mix** will be omitted.

5. Categorical interpretation of Linear Logic

In this section we recall some general material on categorical models of **MLL** and look at the category **CohPS** in this context. We refer the reader to [24] for original references and a more detailed discussion of the categorical semantics.

5.1. Linear Logic and $*$-autonomous categories

It is widely believed that Multiplicative Linear Logic is the logic of $*$-*autonomous categories*. The definition of those is given below.

Definition 15. A monoidal category is a category **C** together with a bifunctor

$$(.) \otimes (.) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

(the tensor product), an object I (the unit) and natural families of isomorphisms

$$a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \tag{23}$$

$$r_A : A \otimes I \to A, \tag{24}$$

$$l_A : I \otimes A \to A, \tag{25}$$

such that the following diagrams commute.

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a"] \arrow[d, "a \otimes id"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a"] & A \otimes (B \otimes (C \otimes D)) \arrow[d, "id \otimes a^{-1}"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a"'] & & A \otimes ((B \otimes C) \otimes D)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "a"] \arrow[dr, "r"'] & & A \otimes (B \otimes I) \arrow[dl, "id \otimes r"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes (A \otimes B) \arrow[rr, "a^{-1}"] \arrow[dr, "l"'] & & (I \otimes A) \otimes B \arrow[dl, "l \otimes id"] \\
& A \otimes B &
\end{tikzcd}$$

A symmetric monoidal category is a monoidal category $(\mathbf{C}, \otimes, I)$ equipped with a natural family of morphisms

$$c_{AB} : A \otimes B \to B \otimes A \tag{26}$$

(the symmetry) such that the following diagrams commute.

$$\begin{tikzcd}
A \otimes B \arrow[r, "c"] \arrow[dr, "id"'] & B \otimes A \arrow[d, "c"] \\
& A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a"] \arrow[d, "c \otimes id"'] & A \otimes (B \otimes C) \arrow[r, "c"] & (B \otimes C) \otimes A \arrow[d, "a"] \\
(B \otimes A) \otimes C \arrow[r, "a"'] & B \otimes (A \otimes C) \arrow[r, "id \otimes c"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

A symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \otimes, I)$ is $*$-autonomous if there exists a full and faithful functor

$$(.)^{\perp} : \mathbf{C} \to \mathbf{C}^{op}$$

together with a natural family of isomorphisms

$$Hom(A \otimes B, C^{\perp}) \to Hom(A, (B \otimes C)^{\perp}). \quad (27)$$

Given a $*$-autonomous category $\mathbf{C} = (\mathbf{C}, \otimes, I, (.)^{\perp})$ one defines new bifunctors $(.)\wp(.)$ (cotensor product) and $(.) \multimap (.)$ (the last one is contravariant on the first variable) in accordance with the syntax of Linear Logic by

$$(.)\wp(.) = ((.)^{\perp} \otimes (.)^{\perp})^{\perp}, \ (.) \multimap (.) = (.)^{\perp}\wp(.),$$

and the object

$$\perp = I^{\perp}.$$

It is easy to see then that

$$(.)^{\perp} \cong (.) \multimap \perp, \quad (28)$$

and

$$A^{\perp\perp} \cong A. \quad (29)$$

Among the examples of $*$-autonomous categories, which arise from mathematical practice, there are various categories of vector spaces (finite dimensional or suitably topologized) and linear operators, the category **Coh** of coherent spaces and the category of **MLL** formulas and proofs.

5.2. Interpretation of **MLL** *in a $*$-autonomous category*

As long as a category **C** is $*$-autonomous it provides a denotational model of **MLL**.

That is, if one fixes the interpretation $p \mapsto [[p]]$ of literals the interpretation $[[.]]$ extends to all **MLL** formulas by

$$[[A^{\perp}]] = [[A]]^{\perp}, \ [[A \otimes B]] = [[A]] \otimes [[B]], \ [[A\wp B]] = [[A]]\wp[[B]].$$

A sequent $\vdash A_1, \ldots, A_n$ is interpreted as the formula $A_1\wp \ldots \wp A_n$.

Interpretation of proofs is as follows.

By definition for each object A of **C** there are isomorphisms

$$\begin{aligned} Hom(A, A) &\cong Hom(I \otimes A, A) \cong Hom(I \otimes A, (A^{\perp})^{\perp}) \\ &\cong Hom(I, (A \otimes A^{\perp})^{\perp}) = Hom(I, A\wp A^{\perp}). \end{aligned}$$

Therefore the identity morphism id_A may be seen as a morphism $I \to A\wp A^{\perp}$, and the Identity axiom $\vdash A, A^{\perp}$ is interpreted by the identity morphism $id_{[[A]]}$.

By similar reasoning one observes that a morphism $I \to \Gamma\wp A$ may be seen as a morphism $\Gamma^{\perp} \to A$, and therefore given two proofs π_1 and π_2 of Γ, A and $A^{\perp}, \Delta$ interpreted as morphisms $[[\pi_1]]$ and $[[\pi_2]]$ one writes them as

$$[[\pi_1]] : [[\Gamma]]^{\perp} \to [[A]], \ [[\pi_2]] : [[A]] \to [[\Delta]]$$

and takes as the interpretation $[[\pi]]$ of the proof π obtained by the Cut rule from π_1 and π_2 the composition

$$[[\pi]] := [[\pi_2]] \circ [[\pi_1]].$$

The Exchange rule is interpreted by means of symmetry and associativity isomorphisms. The Tensor rule is interpreted by taking the tensor product of morphisms (followed by symmetry), and the Par rule does nothing from the semantical point of view.

The invariance of the interpretation under Cut-elimination essentially is equivalent to commutativity of diagrams in Definition 15.

As we mentioned in the previous section most categorical models of **MLL** model also the Mix rule. The following theorem (which we state without a proof) gives a simple characterization of such models.

Theorem 5 ([6]). *A $*$-autonomous category models the Mix rule if there exists a morphism*

$$m : \perp \to I$$

such that the following diagram commutes.

$$\begin{tikzcd} & \perp \otimes \perp \arrow[dl, "m \otimes id"'] \arrow[dr, "id \otimes m"] & \\ I \otimes \perp \cong \perp \arrow[rr, "id"'] & & \perp \cong \perp \otimes I \end{tikzcd}$$

5.3. Compact closed categories

Now we consider a large subclass of $*$-autonomous categories called *compact closed categories*.

Definition 16. A $*$-autonomous category **C** is compact closed if for all objects A, B in **C** holds $(A \otimes B)^{\perp} \cong A^{\perp} \otimes B^{\perp}$.

(Usually one gives another definition and states the definition above as a theorem.)

From the point of view of semantics of Linear Logic, compact closed categories are not very satisfactory since they yield very degenerate models of **MLL** (with tensor and cotensor identified). Yet this is an important class, and in fact most $*$-autonomous categories arising from mathematical practice are compact closed. Among them let us mention the category of finite-dimensional vector spaces and the category **Rel** of sets and relations (with the Cartesian product as tensor and involution $(.)^{\perp}$ being the identity on objects).

There is a general philosophy of dealing with compact closed categories in the context of Linear Logic. In order to provide a sensible model, namely in order to distinguish tensor from cotensor, one equips the objects in the underlying compact closed category with some extra structure (“a coherence”) and accepts only those morphisms which preserve the coherence. One lifts then the $*$-autonomous structure to the new category. The tensor and cotensor products still induce the same operation on the underlying objects of the compact closed category, but are in general distinct on the level of "coherences". Thus one obtains a non-degenerate $*$-autonomous structure. This approach goes back to the original work of Girard who described Linear Logic as the logic of coherent spaces [11], the category of coherent spaces being a refinement of the category of sets and relations, where underlying sets are equipped with coherence relations. Although particular details of such construction may vary the general scheme is quite natural; it has been possible even to give an abstract nonsense formalization of a version of it ([24]). Following the terminology of the paper cited above we call this construction *double glueing* (but we do not necessarily mean by this term the particular implementation formalized in [24]).

Let us look at the category of coherent phase spaces in the light of categorical formalism.

At first note that the symplectic "category" **Wei** while not being a category is compact closed (i.e. all diagrams in Definition 15 make sense and commute). The proof was sketched in Section 3.1. The situation here does not differ in any essential way from the case of the category **Rel** of sets and relations. It is necessary to verify that all relations which play the role of canonical isomorphisms in (23)–(26) are indeed Lagrangian submanifolds of the ambient phase spaces. This however is indeed the case since these relations are graphs of symplectomorphisms.

Now the category **CohPS** of coherent phase spaces is obtained from **Wei** with the use of (a version of) double glueing exactly in the same fashion as the category **Coh** is obtained from **Rel**. The field of contact cones $\tilde{M}$ on a symplectic manifold M plays the role of an "infinitesimal" coherence relation and morphisms between coherent phase spaces indeed preserve the coherence. A remarkable peculiarity in our case is that we start with an object which is not even a category. Nevertheless at the end we come to a category which can be defined in very simple and concrete terms.

Theorem 6. *The category* **CohPS** *is $*$-autonomous and supports the Mix rule. Hence it provides a denotational model for* **MLL + Mix**. *This category is not compact closed hence the corresponding model distinguishes between the Par and Times connectives.*

Proof. In order to see that **CohPS** is $*$-autonomous one needs to check that all natural families of canonical relations determining the structure of a $*$-autonomous category **Wei** are present in **CohPS** as well. This is completely routine. The fact that this category **CohPS** is not compact closed is seen if we consider the identity relation $\Delta_A \subset A^{\perp} \wp A$ for some object A. This relation is never a state in $A^{\perp} \otimes A$.

The Mix rule is supported since the unit object I, which is the single point manifold with all structure trivial, coincides with its dual. $\square$

6. Completeness questions

We turn to a discussion of the completeness questions for the coherent phase space semantics.

In general, in the case of a denotational semantics the questions of completeness are subtler than in the more traditional case of "Tarski-style" semantics whose purpose is to model provability rather than proofs. Typically, given a categorical model of this or that logical calculus it is quite unlikely that *any* morphism in the category under consideration would correspond to some proof. A more reasonable thing to do is to single out some class of morphisms or families of morphisms, which possess some nice property of *uniformity* or *invariance*, and to expect the interpretation to be surjective onto this class. The most widely accepted approach in the present-day literature is to consider formulas as *multi-variant functors* and proofs as *dinatural transformations* between these functors (see the next section) and to establish the completeness of such interpretation. This kind of theorems is what is usually called *full completeness* theorems and we know quite a number of models of **MLL** enjoying the property of full completeness (see [24] for an account). We will call completeness in the above sense *the abstract full completeness*.

However although the abstract full completeness as outlined above seems to be the most reasonable approach to uniformity in the general context in specific cases there may exist other, more specific (and perhaps simpler) notions of uniformity. This is the case, for example, in the context of game-theoretic models of **MLL** where one may replace dinaturality by naturality with respect to embeddings and prove the corresponding completeness result [1].

One of the nice features of the coherent phase space model of **MLL** is that while enjoying the property of the abstract full completeness it remains complete when one chooses some more specific notions of uniformity. To compare with the case of ordinary coherent space semantics which is known to be fully complete [24] consider the following situation. Let A be a coherent space and let us consider in $A \multimap A \times A$ the subset of the form

$$\{(x, x, x) | \ x \in A\}. \tag{30}$$

This set is easily seen to be a clique no matter how one chooses the coherence structure on A and seems to be rather "uniform", i.e. its definition is independent of any choices. Yet this clique does not belong to any dinatural family and certainly is not a denotation of any proof. (A way to exclude such unwanted phenomena in the setting of coherent spaces is the notion of *totality*, see [20,21].)

In the case of coherent phase spaces, however, this example does not cause any uneasiness since the submanifold defined by (30) is not Lagrangian hence it is not even a morphism.

Motivated by this example we start with a specific completeness theorem which is the easiest to formulate and whose proof is both simple and instructive.

6.1. First completeness theorem

Let us say that an interpretation $\phi \mapsto \langle M_\phi, \tilde{\phi}\rangle$ of **MLL** in **CohPS** is *faithful* if for any literal p the manifold M_p is connected and has dimension greater than zero.

Assume that we have chosen a faithful interpretation of **MLL** in **CohPS**.

Let Γ be a unit-free **MLL**-formula and $M = \langle M_\Gamma, \tilde{\Gamma}\rangle$ be its interpretation.

Let $a_1, \ldots, a_n$ be an enumeration of all occurrences of literals in Γ, and let $M_i = M_{a_i}$, $i = 1, \ldots, n$.

For each $i = 1, \ldots, n$ let π_i denote the natural projection of M on the i-th factor.

Definition 17. We say that a submanifold S of M is uniformly defined if it is given by a system of equations of the form

$$\pi_{i_k}(x) = \pi_{j_k}(x) \tag{31}$$

where k ranges over some finite set and for each k there is $\epsilon \in \{1, \perp\}$, such that

$$a_{i_k} = a_{j_k}^{\epsilon} \tag{32}$$

(hence $M_{i_k} = M_{j_k}$ as manifolds).

The family of submanifolds defined by (31) when the interpretation $\phi \mapsto \langle M_\phi, \tilde{\phi}\rangle$ is not fixed is called the uniform family defined by (31).

Our first completeness theorem states that any uniform family of morphisms in **CohPS** comes from a proof in **MLL + Mix**.

Theorem 7. *In notations of Definition 17 let L be a state of M such that the submanifold L is uniformly defined. Then L is a denotation of some proof in* **MLL + Mix**.

Proof. Fix some point $x \in L$ and consider the situation locally. Let

$$x_i = \pi_i(x),$$
$$V = T_x M,\ V_i = T_{x_i} M_i,\ i = 1, \ldots, n,$$
$$l = T_x L.$$

Writing by abuse of notation π_i for $T_x\pi_i$, $i = 1, \ldots, n$, we see that l is a Lagrangian subspace of V given by the same system of equations (31) as L. Assume without loss of generality that all equations in (31) are functionally independent. Let N be the number of these equations.

Claim 1. *For each $k = 0, \ldots, N$ the space l factors into a direct product of Lagrangian subspaces*

$$l \cong l'_k \times l_{i_1 j_1} \times \cdots \times l_{i_k j_k} \tag{33}$$

where

(i) *for each $m = 1, \ldots, k$ it holds that $M_{i_m} = M_{j_m}^{\perp}$, and $l_{i_m j_m}$ is a Lagrangian subspace of $V_{i_m} \times V_{j_m}$ defined by the m-th equation in (31),*

(ii) *l'_k is a Lagrangian subspace of $\prod_{i \neq i_1, j_1, \ldots, i_k, j_k} V_i$, and*

(iii) *the isomorphism (33) is induced by the natural isomorphism*

$$V \cong \prod_{i=1}^{n} V_i.$$

Proof. Proof by induction on k.

If $k = 0$ there is nothing to prove.

Let $k \geq 0$ and assume that the claim is proven for k. Let us write l' for l'_k. Consider the $(k+1)$-th equation in (31). It follows from the factorization (33) and the functional independence of equations in (31) that

$$i_{k+1}, j_{k+1} \notin \{i_1, j_1, \ldots, i_k, j_k\}.$$

So l' is up to a natural isomorphism a Lagrangian subspace of $V'' \times V_{i_{k+1}} \times V_{j_{k+1}}$ where

$$V'' = \prod_{i \neq i_1, j_1, \ldots, i_{k+1}, j_{k+1}} V_i.$$

The image $l_{i_{k+1} j_{k+1}}$ of l' under the projection to $V_{i_{k+1}} \times V_{j_{k+1}}$ is given by the $(k+1)$-th equation in (31). This is possible only if $V_{i_{k+1}} = V_{j_{k+1}}$ as vector spaces. By Lemma 1 the subspace $L_{i_{k+1} j_{k+1}}$ is coisotropic; this may be the case only if the symplectic structure of $V_{i_{k+1}}$ is opposite to that of $V_{j_{k+1}}$. It follows that $M_{i_{k+1}} = M^{\perp}_{j_{k+1}}$ and the subspace $l_{i_{k+1} j_{k+1}}$ is Lagrangian. Applying the last statement of Lemma 1 we finish the proof of the claim. □

Now, consider the factorization (33) when $k = N$. Let $V' = \prod_{i \neq i_1, j_1, \ldots, i_N, j_N} V_i$.

Since the space $l' = l'_N$ is not constrained by any of the equations in (31) it follows that l' is the whole V'. On the other hand by Lemma 1 the space l' is Lagrangian. It follows that $V' = \{0\}$ and, in fact,

$$l \cong l_{i_1 j_1} \times \cdots \times l_{i_N j_N}.$$

Since we have also proven that $M_{i_k} = M^{\perp}_{j_k}$ for each $k = 1, \ldots, N$ it follows from the relation (32) that the correspondence $i_k \mapsto j_k$ establishes a bijection between positive and negative occurrences of literals in Γ.

It follows that we may associate to the system (31) a proof-structure ρ with conclusion Γ by attaching to the tree of subformulas of Γ the Identity links

$$\frac{Id}{a_{i_k} \; a_{j_k}}$$

where $k = 1, \ldots, N$.

Lemma 5. *The proof-structure ρ is a proof-net.*

Proof. Assume that this is not the case.

Let α be a cyclic switching of ρ. Obviously there exist occurrences a_i, a_j of literals in Γ such that $a_i = a_j^{\perp}$, a_i and a_j are connected in ρ by an axiom link, and a_i, a_j meet the cycle in α.

It follows from the factorization (33) that there exists a non-zero vector $v \in l$ such that

$$\pi_k(v) = 0 \; \forall k \neq i, j. \tag{34}$$

We claim that $v \notin \tilde{M}$.

Assume that this is not the case. For any subformula ϕ of Γ let v_ϕ be the image of v under the natural projection $M \to M_\phi$.

Claim 2. *For any subformula ϕ of Γ it holds that $v_\phi \in \tilde{\phi}$.*

Proof. If $v_\phi = 0$ there is nothing to prove. So assume that $v_\phi \neq 0$.

Let $\bar{\rho}$ be the tree of subformulas of Γ. Let us say that a path $\sigma = (\phi_1, \ldots, \phi_m)$ lying in $\bar{\rho}$ is *downward* if for any k, l such that $k < l \leq m$ the formula ϕ_k is a subformula of ϕ_l. Then the proof of the claim is by induction on the downward paths connecting ϕ to Γ in $\bar{\rho}$.

If $\phi = \Gamma$ then there is nothing to prove.

Assume that the immediate successor of ϕ in $\bar{\rho}$ is a formula ψ of the form $\psi = \phi \otimes \phi'$ for some formula ϕ'. By the induction hypothesis $v_\psi \in \tilde{\psi}$ which may be the case only if $v_\phi \in \tilde{\phi}$, $v_{\phi'} \in \tilde{\phi}'$.

Assume that the immediate successor ψ of ϕ is of the form $\psi = \phi \wp \phi'$. We claim that $v_{\phi'} = 0$.

Assume this is not the case. Let us write A for a_i, $A^\perp$ for a_j.

Since by assumption $v_\phi \neq 0$ it follows from (34) that ϕ is connected in $\bar{\rho}$ by a downward path σ either to A or to $A^\perp$. Assume without loss of generality that ϕ is connected by σ to A. By the same reasoning ϕ' is connected by a downward path σ' either to A or to $A^\perp$. Now, since $\bar{\rho}$ is a tree there exists at most one downward path between any of its vertices. Since by assumption ϕ is connected by σ to A it follows that ϕ' is connected by σ' to $A^\perp$, otherwise there would be two distinct paths from A to ψ in $\bar{\rho}$ — one through ϕ, another through ϕ'. Next, again since $\bar{\rho}$ is a tree, there exists a vertex B such that σ and σ' have no intersection above B and coincide below B. It follows then that $B = \psi$.

By definition of a switching either σ' or σ does not lie in the switching α. Let s be the maximal downward path through A lying in α and let s' be the maximal downward path through $A^\perp$ lying in α. Obviously s is a subpath of σ, s' is a subpath of σ', and at least one of them terminates before reaching ψ. But then, since σ and σ' have no intersections above ψ, the paths s and s' have no intersection at all, hence A and $A^\perp$ do not meet the cycle in α. This, however, contradicts our assumption and it follows that $v_{\phi'} = 0$.

Now, by definition $v_\psi \in \tilde{\psi}$ if $0 \neq v_\phi \in \tilde{\phi}$ or $0 \neq v_{\phi'} \in \tilde{\phi}'$. But we have just proven that $v_{\phi'} = 0$ so $v_\phi \in \tilde{\phi}$. The claim is proven. □

Thus we have that both $v_A \in \tilde{A}$, $v_{A^\perp} \in \widetilde{A^\perp}$. Also since v is non-zero it follows from (34) that $v_A \neq 0$. On the other hand by (31) it follows that $v_A = v_{A^\perp}$. Thus $\tilde{A}$ and $\widetilde{A^\perp}$ have a non-zero intersection which is a contradiction. This shows that ρ is indeed a proof-net. □

Since proof-nets are in bijection with **MLL + Mix** proofs we have shown how to associate an **MLL + Mix** proof to the system of equations (31). Obviously the submanifold L is actually the denotation of this proof and the theorem is proven. □

Note that the statement of the theorem is about a *single interpretation* and not about a class of interpretations.

6.2. Abstract full completeness

In this subsection we show that the coherent phase spaces semantics of **MLL** enjoys the property of the abstract full completeness in the sense described in the beginning of the section. The proof has little to do with geometry and is based on the similarity between coherent phase spaces and ordinary coherent spaces.

6.2.1. Dinatural transformations

At first let us fix terminology.

In this section the boldface letters $\mathbf{A}, \mathbf{B}, \ldots$ will stand for tuples

$$(A_1, \ldots, A_n), (B_1, \ldots, B_n), \ldots.$$

Definition 18. Let $\mathbf{M}$ be a category.

A dinatural transformation $\sigma : F_1 \to F_2$ between two multivariant functors

$$F_i : \mathbf{M}^n \times (\mathbf{M}^{op})^n \to \mathbf{M},\ i = 1, 2,$$

consists of a family of morphisms

$$\sigma_{\mathbf{A}} : F_1(\mathbf{A}, \mathbf{A}) \to F_2(\mathbf{A}, \mathbf{A}),$$

where $\mathbf{A}$ ranges over n-tuples of objects of $\mathbf{M}$, such that for any n-tuple $\mathbf{B}$ of objects of $\mathbf{M}$ and n-tuple $\mathbf{f}$ of morphisms

$$f_i : B_i \to A_i,\ i = 1, \ldots, n,$$

the following diagram commutes.

$$\begin{tikzcd}
 & F_1(\mathbf{A},\mathbf{A}) \arrow[r, "\sigma_{\mathbf{A}}"] & F_2(\mathbf{A},\mathbf{A}) \arrow[dr, "F_2(id,\mathbf{f})"] & \\
F_1(\mathbf{B},\mathbf{A}) \arrow[ur, "F_1(\mathbf{f},id)"] \arrow[dr, "F_1(id,\mathbf{f})"'] & & & F_2(\mathbf{A},\mathbf{B}) \\
 & F_1(\mathbf{B},\mathbf{B}) \arrow[r, "\sigma_{\mathbf{B}}"] & F_2(\mathbf{B},\mathbf{B}) \arrow[ur, "F_2(\mathbf{f},id)"'] &
\end{tikzcd}$$

Let $\mathbf{M}$ be a $*$-autonomous category. As we have seen earlier any $*$-autonomous category provides a model for **MLL**. Based on this model one may build a new interpretation where formulas are interpreted by functors rather than by objects of $\mathbf{M}$.

All formulas below are supposed to be built from some fixed finite set $\{p_1, p_1^\perp, \ldots, p_\nu, p_\nu^\perp\}$ of literals. Since in the sequel it will be quite important to distinguish between literals and occurrences of literals let us reserve Latin indices $i, j, \ldots$ for enumerating occurrences and Greek indices $\lambda, \mu, \ldots$ for enumerating literals.

Let Γ be a unit-free **MLL** formula. We assign to Γ a multivariant functor $\Gamma(.,.)$, which we shall denote by the same letter. The assignment is as follows.

Let Γ' be the formula obtained from Γ by replacing each negative literal $p_\mu^\perp$ with a fresh variable q_μ, $\mu = 1, \ldots, \nu$.

Definition 19. In notations as above the functor

$$\Gamma : \mathbf{M}^{\nu} \times (\mathbf{M}^{op})^{\nu} \to \mathbf{M}$$

is defined by the following rule.

For a 2ν-tuple $(\mathbf{A}, \mathbf{B}) = (A_1, \ldots, A_\nu, B_1, \ldots, B_\nu)$ let $[[.]]$ be the interpretation of **MLL** in **M** defined by the assignment

$$[[p_\mu]] = A_\mu,\ [[q_\mu]] = B_\mu,\ \mu = 1, \ldots, \nu. \tag{35}$$

Then $\Gamma(\mathbf{A}, \mathbf{B}) = [[\Gamma']]$.

For a 2ν-tuple of morphisms $(\mathbf{f}, \mathbf{g}) = (f_1, \ldots, f_\nu, g_1, \ldots, g_\nu)$, where

$$f_\mu : A_\mu \to A'_\mu,\ g_\mu : B'_\mu \to B_\mu,\ \mu = 1, \ldots, \nu,$$

the morphism $\Gamma(\mathbf{f}, \mathbf{g}) : \Gamma(\mathbf{A}, \mathbf{B}) \to \Gamma(\mathbf{A}, \mathbf{B})$ is defined by induction on Γ.

If $\Gamma = p$, where p is a positive literal, then $\Gamma(f, g) = f$,
if $\Gamma = p^\perp$, where p is a positive literal, then $\Gamma(f, g) = g^\perp$,
if $\Gamma = \Gamma_1 \otimes \Gamma_2$ then $\Gamma(\mathbf{f}, \mathbf{g}) = \Gamma_1(\mathbf{f}, \mathbf{g}) \otimes \Gamma_2(\mathbf{f}, \mathbf{g})$,
if $\Gamma = \Gamma_1 \wp \Gamma_2$ then $\Gamma(\mathbf{f}, \mathbf{g}) = \Gamma_1(\mathbf{f}, \mathbf{g}) \wp \Gamma_2(\mathbf{f}, \mathbf{g})$.

The actual content of this somewhat lengthy definition is that now a formula Γ is interpreted not by a single object of **M** but by the whole family $\{\Gamma(\mathbf{A}, \mathbf{A}) |\ \mathbf{A} \in \mathbf{M}^n\}$ of objects of **M** obtained as the interpretation of literals varies.

In the same fashion, given a proof σ of a sequent $\Gamma \vdash \Delta$, one associates with σ a family of morphisms $\sigma_{\mathbf{A}} : \Gamma(\mathbf{A}, \mathbf{A}) \to \Delta(\mathbf{A}, \mathbf{A})$ where $\mathbf{A}$ ranges over all ν-tuples of objects of **M** by the following rule.

Define an interpretation $[[.]]$ of **MLL** in **M** by the assignment (35). Then

$$\sigma_{\mathbf{A}} = [[\sigma]].$$

It is well known that under such interpretation **MLL** proofs become dinatural transformations. An abstract full completeness theorem says then that any dinatural transformation in **M** between functors definable by **MLL** formulas comes from an **MLL**-proof.

6.2.2. Abstract full completeness in **CohPS**

Now, let us return to our category **CohPS**.

At first, since **MLL** sequents could conveniently be written as one sided it is sufficient to consider only dinatural transformations of the form

$$\sigma : \mathbf{I} \to \Gamma(.,.),$$

where **I** is the constant functor assigning to any tuple of objects the single-point manifold I and the functor $\Gamma(.,.)$ is definable by some **MLL** formula Γ. For brevity we shall call a dinatural transformation σ of this form a *dinatural family*. We will say that the formula Γ

corresponds to σ. The diagram 18 reduces to the diagram below.

$$\begin{tikzcd}
& \Gamma(\mathbf{A},\mathbf{A}) \arrow[dr, "\Gamma(id,\mathbf{f})"] & \\
\mathbf{I} \arrow[ur, "\Gamma(\mathbf{f},id)"] \arrow[dr, "\Gamma(id,\mathbf{f})"'] & & \Gamma(\mathbf{A},\mathbf{B}) \\
& \Gamma(\mathbf{B},\mathbf{B}) \arrow[ur, "\Gamma(\mathbf{f},id)"'] &
\end{tikzcd}$$

Let us say that a family of manifolds σ is a *disjoint* or *set-theoretic union* of a collection of families $\{\sigma^\alpha\}$ if for any tuple $\mathbf{A}$ the manifold $\sigma_{\mathbf{A}}$ is respectively a disjoint or set-theoretic union $\bigcup_\alpha \sigma^\alpha_{\mathbf{A}}$. Let us say that a family σ *contains* a family σ' if for any tuple $\mathbf{A}$ the manifold $\sigma_{\mathbf{A}}$ contains $\sigma'_{\mathbf{A}}$.

Theorem 8. *Any dinatural family σ in* **CohPS** *is a disjoint union of some collection of families $\{\sigma^\mu\}$ where each σ^μ is a denotation of an* **MLL + Mix** *proof.*

Proof. Proof is essentially combinatorial and consists in some tiny modifications of the proof of the full completeness of **MLL + Mix** in **Coh** due to Tan [24]. The only geometric fact that we shall use is summarized in Note 3. Recall that Note 3 states that for a symplectic manifold M and a finite collection of points $x_1, \dots, x_m \in M$ there exists a Lagrangian submanifold L of M such that $x_1, \dots, x_m \in L$.

At first we want to show that a dinatural family σ does not depend on the coherence structure and is completely determined by the underlying symplectic manifolds.

Lemma 6. *Let $\mathbf{A}, \mathbf{B}$ be ν-tuples of coherent phase spaces. If $A_\alpha = B_\alpha$ as symplectic manifolds for each $\alpha = 1, \dots, \nu$, then $\sigma_{\mathbf{A}} = \sigma_{\mathbf{B}}$ as submanifolds.*

Proof. Let $\mathbf{A}' = (A'_1, \dots, A'_\nu)$ be the ν-tuple of coherent phase spaces such that for each $\alpha = 1, \dots, \nu$ the spaces A_α and A'_α coincide as symplectic manifolds, whereas the field of contact cones determining the structure of a coherent phase space for A'_α is the whole TA'_α. Then it is easy to see that for any $\alpha = 1, \dots, \nu$ the diagonal submanifold

$$\Delta_\alpha = \{(x, x) | \, x \in A_\alpha\}$$

of $A_\alpha \times A_\alpha$ is a morphism between A_α and A'_α in **CohPS**.

Then plugging the vector $\Delta = (\Delta_1, \dots, \Delta_\nu)$ of morphisms $\Delta_\alpha : A_\alpha \to A'_\alpha$ in the diagram in the definition of a dinatural family we immediately get that $\sigma_{\mathbf{A}} = \sigma_{\mathbf{A}'}$. Repeating the argument for $\mathbf{B}$ we get that $\sigma_{\mathbf{B}} = \sigma_{\mathbf{A}'}$ as well. $\square$

In view of this lemma it will be convenient to consider coherent phase spaces which have in some sense trivial structure. So let us fix special names for them.

Let us say that a coherent phase space $A = \langle M_A, A\rangle$ is *zero* if the field of contact cones A contains only the zero section of TM_A. We say that $\langle M_A, A\rangle$ is *total* if A is the whole TM_A.

The usefulness of such objects is illustrated by the following straightforward note.

Note 7. Let A be a zero coherent phase space and B be a total coherent phase space. Then any Lagrangian submanifold L of $A_- \times B$ is a morphism from A to B.

Now, let Γ be the **MLL** formula corresponding to σ and let $a_1, \ldots, a_n$ be an enumeration of all occurrences of literals in Γ.

Let

$$I = \{i \,|\, a_i \text{ is a positive literal}\},\ \ J = \{j \,|\, a_j \text{ is a negative literal}\},$$

and for each $\alpha = 1, \ldots, \nu$ let

$$I_\alpha = \{i \,|\, a_i = p_\alpha\},\ \ J_\alpha = \{j \,|\, a_j = p_\alpha^\perp\}.$$

For each $i = 1, \ldots, n$ let π_i denote the family of projections

$$\pi_i : \Gamma(\mathbf{A}, \mathbf{B}) \to a_i(\mathbf{A}, \mathbf{B}).$$

For any pair of tuples $\mathbf{A}$, $\mathbf{B}$ and point $x \in \Gamma(\mathbf{A}, \mathbf{B})$ write x_i for $\pi_i(x)$, $i = 1, \ldots, n$.

Now let us fix some tuple $\mathbf{A}$ and write L for $\sigma_{\mathbf{A}}$.

Lemma 7. *For any $x \in L$, $i \in I$ there exists some $j \in J$ such that $x_i = x_j$, and $a_i = a_j^\perp$.*

Proof. Assume that there exists $x \in L$, $i \in I$, such that for any $j \in J$ the equality $a_j = a_i^\perp$ implies $x_i \neq x_j$.

Let α be such that $a_i = p_\alpha$.

It is easy to see that the set

$$\Delta' = \{(x', x') |\, x' \in A_\alpha\} - \{(x_i, x_i)\}$$

is a submanifold of $A_\alpha \times A_\alpha$ and is a morphism from A_α to itself in **CohPS**.

So consider the vector of morphisms $\mathbf{f} = (id, \ldots, id, \Delta', id, \ldots, id)$. Plug $\mathbf{f}$ in the diagram in the definition of a dinatural family and note that $x \in \Gamma(id, \mathbf{f}) \circ L$ whereas $x \notin \Gamma(\mathbf{f}, id) \circ L$, which is impossible. $\square$

Thus we have shown that for any tuple $\mathbf{A}$ and any $x \in \sigma_{\mathbf{A}}$ there exists a system of the form

$$x_{i_k} = x_{j_k},\ i_k \in I,\ j_k \in J,\ k \in I, \tag{36}$$

satisfied at x.

It remains to show that this system does not depend either on the tuple $\mathbf{A}$ or on the point $x \in \mathbf{A}$ and that the correspondence $i_k \mapsto j_k$ is a bijection $I \to J$.

Lemma 8. *Let $\mathbf{A}$ be a tuple of coherent phase spaces and let $x \in \sigma_{\mathbf{A}}$. Then there exists a system of equations Σ of the form (36) such that x satisfies Σ and the whole uniform family of submanifolds defined by Σ is contained in σ. Moreover Σ is such that the correspondence $i_k \mapsto j_k$ from (36) establishes a bijection between I and J.*

Proof. Let $\mathbf{B}$ be some tuple of coherent phase spaces and pick a tuple of points $\mathbf{y}$, $y_j \in a_j(\mathbf{B}, \mathbf{B})$, $j \in J$.

By Note 3 there exist Lagrangian submanifolds $f_1, \ldots, f_\nu$ of $(B_1)_- \times A_1, \ldots, (B_\nu)_- \times A_\nu$ respectively such that for each α and $j \in J_\alpha$ the manifold f_α contains (y_j, x_j). Removing, if necessary, a finite number of points from f_α we may also achieve that for each $\alpha = 1, \ldots, \nu$, $i \in I_\alpha$, $j \in J_\alpha$, it holds that

$$(y_j, x_i) \in f_\alpha \text{ iff } x_i = x_j. \tag{37}$$

Without loss of generality we assume that all coherent phase spaces $A_1, \ldots, A_\nu$ are total and $B_1, \ldots, B_\nu$ are zero. Then by Note 7 for each $\alpha = 1, \ldots, \nu$ the canonical relation f_α is a morphism from B_α to A_α.

Now let us look at our diagram in the definition of a dinatural family.

Let $L = \Gamma(\mathbf{f}, id) \circ \sigma_{\mathbf{B}} = \Gamma(id, \mathbf{f}) \circ \sigma_{\mathbf{A}}$.

Chasing the upper leg of the diagram we see from (37) that L contains a point z defined by:

$$\pi_i(z) = x_i,\ i \in I,\ \pi_j(z) = y_j,\ j \in I. \tag{38}$$

Chasing the lower leg of the diagram we see that there exists a point $y \in \sigma_{\mathbf{B}}$ such that y is sent to z by $\Gamma(\mathbf{f}, id)$ i.e. such that $(y, z) \in \Gamma(\mathbf{f}, id)$. (Obviously $\pi_j(y) = \pi_j(z) = y_j$ where y_j is the j-th element of the tuple $\mathbf{y}$ so our notation is consistent). Note that for any $\alpha = 1, \ldots, \nu$, $i \in I_\alpha$, there holds

$$(y_i, x_i) \in f_\alpha. \tag{39}$$

By Lemma 7 there exist two systems of equations Σ and Σ' of the form (36) satisfied at x and y respectively.

Claim 3. *If all entries x_i where i ranges over I are distinct then y satisfies Σ.*
If all entries y_j where j ranges over J are distinct then x satisfies Σ'.

Proof. The set of entries $\{y_i | \ i \in I\}$ is contained in the set of entries $\{y_j | \ j \in J\}$. Now the claim is immediate from (39) and (37). □

So, assume, at first, that our x is such that all entries x_i, $i \in I$, are distinct. Then since both the tuple $\mathbf{B}$ and the tuple $\mathbf{y}$ were arbitrary it follows from the claim above that σ contains the whole uniform family defined by Σ. Note also that in this case Σ induces a well-defined function $J \to I$. If the tuple $\mathbf{y}$ was chosen such that all y_j, $j \in J$, are distinct then since y satisfies Σ it follows that Σ induces also a well defined map $I \to J$ hence Σ induces a bijection $I \to J$.

Assume now that not all entries x_i, $i \in I$ are distinct. Then, since the tuple $\mathbf{y}$ was arbitrary, we may assume that all entries y_j, $j \in J$, are distinct. Obviously the previous argument may be repeated with J interchanged with I to show that there exists a system Σ' satisfied at y such that Σ' induces a bijection $I \to J$ and the whole uniform family defined by Σ' is contained in σ. But by the claim above x satisfies Σ' as well. □

We have proven that σ is a set-theoretic union of some collection of uniform families $\{\sigma^\mu\}$. To see that this is actually a disjoint union observe that it follows from the last statement of the previous lemma that for any tuple $\mathbf{A}$ the manifold $\sigma^{\mu}_{\mathbf{A}}$ has the same dimension as $\sigma_{\mathbf{A}}$. Namely both of them have dimensions equal to half of that of $\Gamma(\mathbf{A},\mathbf{A})$, the first by properties of the defining system of equations, the second because it is Lagrangian. Thus $\sigma^{\alpha}_{\mathbf{A}}$ is an open neighborhood of $\sigma_{\mathbf{A}}$. But since $\sigma^{\alpha}_{\mathbf{A}}$ is also closed it follows that $\sigma^{\alpha}_{\mathbf{A}}$ is a union of some collection of connected components of $\sigma_{\mathbf{A}}$.

It remains to show that for each μ the family σ^{μ} is a denotation of an **MLL + Mix** proof.

We have shown that the system Γ of equations of the form (36) which defines a uniform family σ^{μ} establishes a bijection between the set of positive and negative occurrences of literals in Γ. So it is clear how to associate with σ^{μ} a proof-structure. Proof of the fact that this proof-structure is actually a proof-net is the same as in Lemma 5. $\square$

Note that Theorem 8 does not hold for the analogous category where manifolds and submanifolds are replaced with vector spaces and subspaces and fields of contact cones are replaced simply with contact cones. In fact the family σ_V of Lagrangian subspaces of $V \times V_-$, where $\sigma_V = \{(v,-v) \mid v \in V\}$, is a natural transformation of type $V \multimap V$.

6.3. Local full completeness

We have seen that the symplectic "category" **Wei** is $*$-autonomous so it makes sense to speak about an interpretation of **MLL** in **Wei**. Of course such an interpretation is defined exactly as in the case of **CohPS** — we just forget the extra structure of a field of contact cones. In fact, as one may note, our interpretation in **CohPS** is actually an interpretation in **Wei** since it does not depend on this extra structure. A remarkable thing is that if a morphism (i.e. a Lagrangian submanifold) in **Wei** comes from an **MLL** proof then it remains a morphism in **CohPS** for any choice of the "coherence". Our third completeness theorem says that the converse is also true.

More precisely the statement is as follows. Assume that we have chosen a faithful interpretation $\phi \mapsto M_\phi$ of **MLL** in **Wei**.

Theorem 9. *Let Γ be a unit-free* **MLL** *formula and let $M = M_\Gamma$ be its interpretation. Assume that L is a closed connected (and nonempty) Lagrangian submanifold of M such that for any choice of fields of contact cones $\tilde{p}$ on M_p, where p ranges over positive literals, L is a state of $\langle M, \tilde{\Gamma}\rangle$. (Here $\tilde{\Gamma}$ is obtained by extending inductively the interpretation $p \mapsto \langle M_p, \tilde{p}\rangle$ to all formulas.)*

Then L is a denotation of a proof in **MLL + Mix**. *(If L is not connected then the statement above holds for each of its connected components. If L is not closed then the statement holds locally.)*

Proof. Let $a_1,\ldots,a_n$ be an enumeration of all occurrences of literals in Γ.

For each $i, j = 1,\ldots,n$ where $i \neq j$ let us put $M_i = M_{a_i}$ and denote by π_{ij}, π_i the natural projections

$$\pi_i : M \to M_i,\ \pi_{ij} : M \to M_i \times M_j.$$

In general for a subformula ϕ of Γ denote by π_ϕ the obvious projection

$$\pi_\phi : M \to M_\phi.$$

Now lets us fix a point $x \in M$ and let $x_i = \pi_i(x),\ i = 1, \ldots, n$.

Let $V = T_x M,\ V_i = T_{x_i} M_i,\ i = 1, \ldots, n$.

Abusing the notation let us write $\pi_i,\ \pi_{ij}$ and π_ϕ for $T_x\pi_i,\ T_x\pi_{ij},\ T_x\pi_\phi$ respectively, where $i = 1, \ldots, n$ and ϕ is any subformula of Γ. For $v \in V$ write v_i and v_ϕ for $\pi_i(v)$ and $\pi_\phi(v)$ where i and ϕ are as above.

Let $l = T_x L$.

We start with a simple observation.

Note 8. For any nonzero $v \in l$ there exist i, j such that $a_i = a_j^\perp$, $x_i = x_j$ (hence $V_i = (V_j)_-$) and $v_i = \lambda v_j$ for some $0 \neq \lambda \in \mathbf{R}$.

Proof. Assume that the statement does not hold.

Then there exists $0 \neq v \in l$ such that $\forall i, j$ $a_i = a_j^\perp$ and $x_i = x_j$ imply that either $v_i = 0$, or $v_j = 0$, or v_i and v_j are linearly independent.

For each positive literal p define the field of contact cones $\tilde{p}$ on M_p by

$$\tilde{p} = \{t\sigma \in TM_p \mid \sigma = v_j \text{ for some } j \text{ s.t. } a_j = p^\perp,\ t \in \mathbf{R}\}. \tag{40}$$

Now all positive literals being assigned coherent phase spaces this assignment extends to an assignment $\phi \mapsto \tilde{\phi}$ for all formulas. It is not hard to see however that $v \notin \tilde{\Gamma}$.

Let us prove that for any subformula ϕ of Γ there holds

$$v_\phi \notin \tilde{\phi} \text{ or } v_\phi = 0. \tag{41}$$

by induction on the subformula.

If ϕ is a positive literal, say a_i, then $v_\phi = v_i$. If $0 \neq v_i \in \tilde{a}_i$ then $v_i = tv_j$ for some j, t, such that $a_i = a_j^\perp$, $t \neq 0$. This however contradicts the hypothesis of the lemma.

If ϕ is a negative literal, say $a_i^\perp$, then by definition $v_\phi = v_i \in \tilde{a}_i$ hence $v_\phi \notin \tilde{\phi}$ unless $v_\phi = 0$.

The cases when $\phi = \phi_1 \otimes \phi_2$ or $\phi = \phi_1 ⅋ \phi_2$ are trivial. $\square$

The note above has an important consequence.

Note 9. All projections π_i, $i = 1, \ldots, n$ restricted to l are surjective.

Proof. Fix some i. Assume that $l_i = \pi_i(l)$ is not the whole V_i. Then there exists some nonzero $u \in orth(l_i)$. By Lemma 1 the space l contains a vector v all of whose components $v_1, \ldots, v_n$ are zero except v_i which is equal to u. But by the note above this cannot happen unless $u = 0$. $\square$

Now let us proceed to the proof of the theorem.

Let N be the number of positive occurrences of literals in Γ. The essence of the proof of the theorem consists in proving the following.

Claim 4. *For any $k = 0, \ldots, N$ there exist distinct integers $i_1, \ldots, i_k \leq n$ satisfying the following property. For each $i = i_1, \ldots, i_k$ the literal a_i is positive and there exists a unique $j = j(i)$ such that*

(i) *$a_i = a_j^\perp$, and $x_i = x_j$ (hence $V_i = (V_j)_-$),*

(ii) *$l_{ij} = \pi_{ij}(l)$ is a Lagrangian subspace of $V_i \times V_j$.*

Moreover

$$l \cong l' \times l_{i_1 j_1} \times \dots l_{i_k j_k} \tag{42}$$

where l' is a Lagrangian subspace of $\prod_{i \neq i_1,\dots,i_k} V_i$, and the isomorphism above is induced by a permutation of factors in $V = V_1 \times \cdots \times V_n$.

Proof. Proof by induction on k.

For $k = 0$ there is nothing to prove.

Assume that the claim is proven for a given k. In particular let the factorization (42) be given.

Now if $l' = \{0\}$ we are done. So assume that this is not the case. Then with the use of Notes 8 and 9 the following can be proven.

Lemma 9. *There exist $i, j \in \{1,\dots,n\} - \{i_1, j_1, \dots, i_k, j_k\}$, $i \neq j$, and $\lambda \neq 0$ such that for any $v \in l'$*

$$v_i = \lambda v_j. \tag{43}$$

The proof of this fact is a rather technical exercise in linear algebra and will be postponed.

Note that the identity (43) implies that $x_i = x_j$ since two vectors tangent to a manifold can be equal only if they are tangent at the same point.

So let $i, j \neq i_1, j_1, \dots, i_k, j_k$ be such that (43) holds and let $l_{ij} = \pi_{ij}(l)$. By (42) we have that $l_{ij} = \pi_{ij}(l')$. It follows that $l_{ij} = \{(v, \lambda v) | v \in V_i\}$ for some λ. By Lemma 1 the space l_{ij} is coisotropic. But this may be the case only if $|\lambda| = 1$ and $V_i = (V_j)_-$ hence $a_i = a_j^{\perp}$. Then l_{ij} is Lagrangian. Put $i_{k+1} = i$, $j_{k+1} = j$. The last statement of Lemma 1 finishes the proof of the claim for $k + 1$. □

It follows from the proof of the claim above that in fact

$$l \cong \prod l_{ij(i)}$$

hence the correspondence $i \mapsto j(i)$ is bijective. Note that we proved as well that each subspace l_{ij} is either the diagonal or antidiagonal (i.e. the graph of multiplication by -1) subspace of $V_i \times V_j$. The possibility of l_{ij} being the antidiagonal is of course an unwanted feature (and we shall prove in a minute that this never happens). The nature of this phenomenon consists in the fact that multiplication by -1 is a natural symplectomorphism in the category of symplectic vector spaces. This symplectomorphism depends however on the algebraic structure which is absent in the case of manifolds. Essentially this means that if we worked in the setting of symplectic vector spaces, i.e. symplectic manifolds with *fixed* vector space structures, and required each field of contact cones to be constant (this requirement itself depends on the choice of coordinates and vector space structure because the notion of constancy is different in different coordinate systems and does not make sense at all in the absence of a global trivialization of the tangent bundle) we would be able to obtain our completeness result only "up to a sign".

Let us return to the proof of our theorem. Since $x_i = x_{j(i)}$ we have proven the following: Each point $x \in L$ satisfies a system of equations of the form

$$\pi_i(x) = \pi_j(x) \tag{44}$$

where $a_i = a_j^\perp$ and the correspondence $i \mapsto j$ establishes a bijection between positive and negative occurrences of literals in Γ.

Let us prove that the system (44) does not depend on the choice of a point $x \in L$.

Indeed, there is only a finite number of systems of the form (44) and each such system defines a submanifold of M of dimension equal to $\dim L$. So L lies in the union of a finite collection $L_1, \ldots, L_m$ of submanifolds of M whose dimensions are equal to that of L. Considering the situation locally we may assume that all manifolds $M_1, \ldots, M_\mu$, which are interpretations of literals $p_1, \ldots, p_\mu$, and the whole ambient manifold M are in fact vector spaces. Then all submanifolds $L_1, \ldots, L_m$ are distinct vector subspaces of M, all of the same dimension. For any point $x \in L$ the tangent space $T_x L$ lies in the union $\bigcup_{i=1}^{m} L_i$. It is easy to see that then the space $T_x L$ coincides with L_i for some $i = 1, \ldots, m$. We claim that for all points $x' \in L$ in a neighborhood of x it holds that $T_{x'} L = L_i$. If this is not the case then there exists a sequence $\{x_n\}$ of points in L converging to x with $T_{x_n} L = L_{i_n} \neq L_i$. Consider the *Lagrangian Grassmanian bundle* $\Lambda(M)$. This is a bundle over M whose fiber over a point $y \in M$ is the manifold $\Lambda(T_y M)$ of all Lagrangian subspaces of $T_y M$. Since L is a smooth submanifold of M the sequence $\{T_{x_n} L\}$ should converge to $T_x L$ in $\Lambda(M)$. Since $\Lambda(M)$ is locally trivial, i.e. for a neighbourhood U of x we have $\Lambda(U) \cong U \times \Lambda(T_x M)$, this convergence implies that the sequence $\{L_{i_n}\}$ converges to $T_x L = L_i$ in $\Lambda(T_x M)$. However this sequence belongs to a finite set $\{L_1, \ldots, L_m\}$ hence (for n sufficiently large) we have $L_{i_n} = L_i$.

Thus for all $x' \in M$ in a neighborhood U of x the tangent space $T_{x'} L$ coincides with L_i hence the whole manifold L coincides with L_i on U as well.

Now let y be another point of L. By assumption the manifold L is connected hence there is a path $\phi : [0, 1] \to L$ with the endpoints $\phi(0) = x$ and $\phi(1) = y$. Note that $\mathrm{Im}\phi$ is compact. It follows from the reasoning above that we may cover $\mathrm{Im}\phi$ by a finite system of open neighborhoods $U_1, \ldots, U_k$ such that on each U_j, $j = 1, \ldots, k$, the manifold L coincides with one of $L_1, \ldots, L_m$. Since these neighborhoods will have overlaps we deduce by induction on k that this L_i is the same along the whole path ϕ. Since y was arbitrary we see that L coincides with L_i at all of its points.

A formal justification of various steps in the proof above is a completely routine exercise in differential geometry and is left to the reader.

Thus L belongs to the solution set L' of some system of the form (44) and since $\dim L = \dim L'$ it follows that L is an open subset of L'. Finally since by assumption L is closed we get that $L = L'$. This shows that L belongs to some uniform family and the theorem follows from (31). (Actually we need only Lemma 5 since it is clear from the above how to associate to L a proof-structure). $\square$

Note that the local full completeness theorem of this section differs substantially from Theorem 8 of the previous section. Theorem 8 is global in the sense that its very formulation makes sense only if one considers the whole category **CohPS** (or, maybe, a sufficiently large subcategory); one has to work with all possible interpretations of the language. The local full completeness on the contrary is formulated in terms of a single tuple of symplectic manifolds. In particular the global full completeness does not follow from the local one; given a dinatural family σ each member $\sigma_{\mathbf{A}}$ of σ satisfies the conditions of the local full completeness theorem and is, consequently, a denotation of the proof. However the independence of this proof on the tuple $\mathbf{A}$ follows from global features of σ; and eventually a global analysis turns out to be more efficient in this context.

6.3.1. Proof of Lemma 9

At first let us fix some terminology.

Definition 20. Let V, U be vector spaces and L, M be linear operators

$$L : V \to U,\ M : V \to U.$$

A vector $v \in V$ is an *eigenvector* for the (unordered) pair (L, M) if there exist scalars λ, μ such that $(\lambda, \mu) \neq 0$ and

$$\lambda L v + \mu M v = 0. \tag{45}$$

The point $(\lambda : \mu) = \{(t\lambda, t\mu) |\ t \in \mathbf{R} - \{0\}\}$ of the projectivization $\mathbf{RP}^1$ of $\mathbf{R}^2$ is an eigenvalue of (L, M) corresponding to v. (Eigenvalues in general are not uniquely determined by eigenvectors; if $Lv = Mv = 0$ then any point of $\mathbf{RP}^2$ is an eigenvalue of (L, M) corresponding to v.)

Let us establish a couple of properties of eigenvectors and eigenvalues.

Lemma 10. *In notations as above let $v_1, \ldots, v_n$ be eigenvectors of (L, M) such that not all corresponding eigenvalues are equal. Assume that there exists a vector $0 \neq v \in V$ such that $v = \sum s_i v_i$ for some scalars s_i where $s_i \neq 0$ for all $i = 1, \ldots, n$ and v is also an eigenvector of (L, M). Then vectors $Lv_1, \ldots, Lv_n$ are linearly dependent.*

Proof. If for some $i = 1, \ldots, n$ it holds that $Lv_i = 0$ then we are done. So let us assume that $Lv_i \neq 0$ for all i.

Let $(\lambda_1 : \mu_1), \ldots, (\lambda_n : \mu_n)$ be some eigenvalues corresponding to $v_1, \ldots, v_n$ respectively. It follows then that $\mu_i \neq 0$ for any $i = 1, \ldots, n$. Indeed if $\mu_i = 0$ for some i then since by assumption $Lv_i \neq 0$ we have from (45) that $\lambda_i = \mu_i = 0$ which contradicts the definition of an eigenvector.

Thus for any $i = 1, \ldots, n$

$$M v_i = -\frac{\lambda_i}{\mu_i} L v_i. \tag{46}$$

Now let $(\lambda : \mu)$ be some eigenvalue corresponding to v. We have

$$\lambda L v + \mu M v = \sum \frac{s_i}{\mu_i} (\lambda \mu_i - \lambda_i \mu) L v_i = 0. \tag{47}$$

By hypothesis $s_i \neq 0$ for all $i = 1, \ldots, n$; so linear independence of $Lv_1, \ldots, Lv_n$ implies that

$$\lambda \mu_i - \lambda_i \mu = 0,\ i = 1, \ldots, n, \tag{48}$$

i.e. that for each i the pair (λ_i, μ_i) is a scalar multiple of (λ, μ). But that means that $(\lambda_i : \mu_i) = (\lambda : \mu)$ for all $i = 1, \ldots, n$, which contradicts our hypothesis. □

Lemma 11. *In the same notations if v_1, v_2 are eigenvectors of (L, M) and for some scalars s_1, s_2, such that $s_i \neq 0$, $i = 1, 2$, the vector $v_3 = s_1 v_1 + s_2 v_2$ is also an eigenvector of (L, M) then any linear combination of v_1 and v_2 is an eigenvector of (L, M).*

Proof. Let $(\lambda_i : \mu_i)$, $i = 1, 2, 3$, be some eigenvalues of (L, M) corresponding to v_i, $i = 1, 2, 3$, respectively. If for some pair i, j, such that $i \neq j$, it holds that $(\lambda_i : \mu_i) = (\lambda_j : \mu_j)$ then the statement holds by linearity and homogeneity of (45). So assume that all $(\lambda_i : \mu_i)$, $i = 1, 2, 3$, are distinct.

By the previous lemma the vectors Lv_1 and Lv_2 span a ≤ 1-dimensional subspace of U. Let us denote this subspace by l.

By symmetry the span m of Mv_1 and Mv_2 is also at most one dimensional.

Furthermore at least for some $i = 1, 2, 3$ both Lv_i and Mv_i are nonzero. Indeed if for any $i = 1, 2, 3$ it holds that either $Lv_i = 0$ or $Mv_i = 0$ then there exist i, j, $i \neq j$, such that $Lv_i = Lv_j = 0$ or $Mv_i = Mv_j = 0$. But then either $(0 : 1) = (\lambda_i : \mu_i) = (\lambda_j : \mu_j)$ or $(1 : 0) = (\lambda_i : \mu_i) = (\lambda_j : \mu_j)$, both possibilities contradicting our assumption. Thus there exists i such that Lv_i and Mv_i are nonzero hence λ_i and μ_i are nonzero. Then $0 \neq \lambda_i L v_i = -\mu_i M v_i \in l \cap m$. Thus l and m have a nonzero intersection and since each of them is one dimensional it follows that they coincide. Then the claim immediately holds. □

Now we are ready to prove our lemma. In fact we will prove a more general statement.

Lemma 12. *Let V, $V_1, \ldots, V_k$ be vector spaces and $L_1, \ldots, L_k, M_1, \ldots, M_k$ be linear operators*

$$L_i : V \to V_i, \;\; M_i : V \to V_i,$$

such that

$$\dim \mathrm{Im} L_i > 1, \;\; \dim \mathrm{Im} M_i > 1, \;\; i = 1, \ldots, k. \tag{49}$$

Assume that any vector $v \in V$ is an eigenvector for some pair (L_i, M_i). Then in fact there exists i such that all vectors in V are eigenvectors of (L_i, M_i) with the same eigenvalue.

Proof. Proof by induction on k.

Let $k = 1$. Then there is only one pair (L, M). If there exists one eigenvalue of (L, M), which corresponds to all vectors in V, then we are done. So assume that there exist $v_1, v_2 \in V$, such that all corresponding eigenvalues are distinct. (Note that then for each $i = 1, 2$ either $Lv_i \neq 0$ or $Mv_i \neq 0$.)

Let us show that either Lv_1 or Lv_2 is in fact zero.

Indeed, by Lemma 10 vectors Lv_1, Lv_2 lie in a one-dimensional subspace of U. Let us denote this subspace by l. Now let v be another vector in V which is distinct from v_1, v_2. Obviously at least for one $i = 1, 2$ the pair $\{v_i, v\}$ satisfies the conditions of Lemma 10. Assume without loss of generality that $i = 1$. We have that the span l' of Lv_1, Lv is also at most one dimensional and $0 \neq Lv_1 \in l \cap l'$. It follows that the subspaces l' and l coincide and that in particular $Lv \in l$. Since v was arbitrary we see that $\mathrm{Im}L \subset l$ hence $\mathrm{Im}L$ has dimension at most 1 which contradicts (49).

By symmetry we have also that either Mv_1 or Mv_2 is zero. Assume without loss of generality that $Lv_1 = Mv_2 = 0$ (the possibility $Lv_i = Mv_i = 0$ is excluded). This means that v_1 has the only eigenvalue $(1 : 0)$ and v_2 has the only eigenvalue $(0 : 1)$. Clearly we may repeat the argument and deduce that for any vector $v \in V$, such that $Lv \neq 0$ or $Mv \neq 0$, the only eigenvalue of v is either $(0 : 1)$ or $(1 : 0)$. Now let $v = v_1 + v_2$. We have that $Lv = Lv_2 \neq 0$ and $Mv = Mv_1 \neq 0$. By assumption v is also an eigenvector. So for some $(\lambda, \mu \neq 0)$ it holds that

$$0 = \lambda L(v_1 + v_2) + \mu M(v_1 + v_2) = \lambda L v_2 + \mu M v_1.$$

It follows that both λ and μ are nonzero since both Lv_2 and Mv_1 are nonzero. In particular $(\lambda : \mu) \neq (0 : 1)$ and $(\lambda : \mu) \neq (1 : 0)$. This gives us a contradiction.

Now let $k > 1$.

We may assume that there exists at least one eigenvector v of the pair L_1, M_1. If no such v exists then the hypothesis of the lemma holds for a smaller number of pairs (L_i, M_i), and the statements holds by induction.

Assume also that there exists some $u \in V$ which is not an eigenvector of (L_1, M_1). If no such u exists then again the statement holds by induction.

Consider the affine line $A = \{tv + (1 - t)u |\ t \in \mathbf{R}\}$.

We claim that the pair (L_1, M_1) has at most two distinct eigenvectors in A. Indeed A is spanned by any pair of its distinct elements. So if there exist at least three distinct eigenvectors of (L_1, M_1) in A then Lemma 11 applies and any element of A is an eigenvector of (L_1, M_1). Since $u \in A$ this is impossible.

Now since A has infinite cardinality there exists at least one $i = 1, \ldots, k$ such that the set of eigenvectors of (L_i, M_i) in A is infinite. By the observation above $i \neq 1$. But then Lemma 11 applies again and v is an eigenvector of (L_i, M_i).

Since v was arbitrary we have proven that any eigenvector of (L_1, V_1) is also an eigenvector of some (L_i, M_i), $i \neq 1$. But then the hypothesis of the lemma holds for $(L_2, M_2), \ldots, (L_k, M_k)$ and the statement holds by induction. □

Proof of Lemma 9 is now immediate.

It follows from (42) that for each $i \neq i_1, j_1, \ldots, i_k, j_k$ the image $\pi_i(l')$ coincides with $\pi_i(l)$. By Note 9 the subspace $\pi_i(l')$ is the whole V_i and since the latter is a nontrivial symplectic vector space its dimension is at least 2. Combining this with Note 8 and the previous lemma we obtain the desired result. □

7. Discussion

In this section we make a few remarks on the extensions of the interpretation to other fragments of Linear Logic.

7.1. Multiplicative neutrals

The interpretation of multiplicative neutrals $\perp$ and **1** has been used implicitly throughout the paper. The category **CohPS** being $*$-autonomous has a unit object I, namely the single-point manifold (see Section 3.1). This object provides an interpretation for both $\perp$ and **1**, the interpretation thus being degenerate. This degeneracy is expected since the coherent phase space semantics models the Mix rule.

7.2. Additives

The additive connectives of Linear Logic are $\oplus$ (*plus*) and & (*with*). Their categorical meaning is that of the coproduct and the product respectively. Coherent phase spaces provide a natural interpretation for the additives, but this is not completely satisfactory as the interpretation is degenerate.

In the category **Rel** of relations there exist products and coproducts, both operations being given by the disjoint union. This biproduct passes to the symplectic "category" **Wei** and to our category **CohPS** as one readily checks. Namely the biproduct of two coherent phase spaces is the coherent phase space obtained by taking the disjoint union of the underlying manifolds with all other structure defined in the obvious way; the biproduct of two morphism is simply the disjoint union of the corresponding immersions. Thus our model has degenerate additives. The main problem with the additives is that the coherent phase space model is concerned only with local phenomena whereas the disjoint union is a global operation. Perhaps the additive fragment is precisely the point which marks the limits of the "semi-classical" approach, and in order to give a satisfactory interpretation one has to go indeed to the realm of Hilbert spaces and the full machinery of quantum theory. (Typically it is tempting to take linear combinations of Lagrangian submanifolds and assign norms to them, which would eventually involve all algebraic and analytic structure with all mathematical difficulties.) Nevertheless we believe that geometric ideas of our model may shed some light on how this "quantum semantics" could be developed.

7.3. Exponentials

The interpretation of the exponential fragment is certainly one of the most interesting challenges for any model of Linear Logic. In fact after some reasonable modifications of the model the coherent phase spaces provide, at least on the informal, "physicist's" level of rigor, an interpretation for exponentials. This interpretation however is quite ill-justified from the point of view of rigorous mathematics although after some severe restrictions on the objects a certain fragment can be treated rigorously.

(This informal model for exponentials is somewhat reminiscent of the quantization of field theory. The quantization of field theories is done by physicists with varying success and levels of rigor by pretending that a field theoretic system is in fact a mechanical system, but with infinitely many degrees of freedom; then one attempts to apply methods of quantization of mechanic to the infinite-dimensional context, see for example [9]. The mathematical treatment however is often problematic, in particular, due to the fact that infinite-dimensional differential geometry simply does not yet exist at any satisfactory level. The usage of methods of geometric quantization and semi-classical approximation in the context of quantum field theory is in general highly heuristic and ill-justified yet these methods play an important role in today mathematical physics, for example in such fashionable topic as topological quantum field theory, see [2].)

The model for exponentials is based on quite advanced machinery of geometric quantization, connections and contact structures and it is hopeless to introduce all these sophisticated concepts in a brief concluding section. Let us mark however some starting points of the construction. A detailed and, hopefully, mathematically rigorous presentation is still in preparation.

The exponential connectives ! (*bang*) and ? (*why not*) of Linear Logic are responsible for the reuse of an argument; typically the formula $!A \multimap B$ means that the hypothesis A can be used more than once or, perhaps, not used at all in the implication to B. One of the main features of the bang ! is a possibility of duplication, which is expressed by the natural morphism $c_A :!A \to !A \otimes !A$ on the semantical level and by the rule of *Contraction*

$$\frac{!A, !A \vdash \Gamma}{!A \vdash \Gamma}$$

on the level of syntax. We want to find a corresponding structure in the symplectic "category".

Recall that there is a *cotangent lift* functor $T^*(.)$ from the category **Man** of smooth manifolds to the category of coherent phase spaces (see Note 6). In the category **Man** of (infinitely!) smooth manifolds we have the following: a function $f(.,.)$ of two variables, which is separately infinitely smooth in each variable, is also jointly infinitely smooth; in particular, the diagonalization $x \mapsto f(x, x)$ is smooth (this holds because partial derivatives of all orders are continuous). Roughly speaking this means that the category **Man** models the rule of contraction. It is natural to expect that the exponentiation should respect functoriality of the cotangent lift. The diagonalization

$$f(x, y) \mapsto f(x, x)$$

of a function corresponds to the addition of partial derivatives

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \mapsto \left(\frac{\partial f}{\partial x}(x, x) + \frac{\partial f}{\partial y}(x, x)\right)dx$$

and differentials live in the fibers of the cotangent bundle. Thus on the level of cotangent lifts diagonalization corresponds to addition in the fibers of the cotangent bundle. The latter addition is represented in the symplectic "category" **Wei** by its graph

$$c = \{((x, \phi_x + \psi_x), (x, \phi_x), (x, \psi_x))\}. \tag{50}$$

One may check that the cotangent lift of $f(x, x)$ is obtained by composing the cotangent lift of $f(x, y)$ with the relation c above. (Note that c itself is the cotangent lift of the diagonal map $x \mapsto (x, x)$.)

Thus if we want to respect functoriality of the cotangent lift it is natural to assume the following: If A is in the image of the cotangent lift functor, i.e. if A is a coherent cotangent bundle, then $!A$ should coincide with A and the morphism c_A should be the graph of the addition in the fibers of A as in (50).

The problem with the morphism c in (50) is that c is not natural even if we restrict the class of objects to coherent cotangent bundles. A simple way to see this is as follows. The family in (50) depends on the vector space structure on the fibers of the cotangent bundle.

If this morphism were natural then for each cotangent bundle the vector space structure on the fibers would be determined by the symplectic structure and the fibration (since this is the only data encoded in the definition of a coherent cotangent bundle). Remarkably enough the *affine* space structure on the fibers of a cotangent bundle indeed can be recovered from the symplectic structure and the fibration (see [27], 4.7.). However the vector space structure depends on the location of the zero-section. And the only characteristic of the zero-section, which can be stated in terms of the symplectic structure and the fibration, is that the zero-section is a Lagrangian submanifold transversal to the fibers. Therefore various choices of vector space structures on fibers are parametrized by such submanifolds. Lagrangian submanifolds of the cotangent bundle T^*Q, which are transversal to the fibers, are graphs of closed 1-forms defined on the base Q. Thus the morphism c in (50) is natural up to a "gauge transformation" induced by a closed 1-form.

A solution to the problem of this "gauge ambiguity" exists and consists in a modification of the category **CohPS**. One should further pursue the interpretation of Lagrangian submanifolds as semi-classical states and equip each Lagrangian submanifold σ with a *phase function* defined on σ (see Section 1.3.3). (The ambient spaces should be equipped with an extra phase coordinate and they become *prequantum bundles* over symplectic manifolds, a standard structure in geometric quantization.) Phase functions encode the extra degrees of freedom corresponding to the location of the zero-section. In general, for a coherent phase space A the space $!A$ is defined as the coherent cotangent bundle over the "manifold" Q_A of (closed and connected) states of A. (Certainly this needs a more accurate mathematical definition and justification!)

A quasi-physical explanation is as follows. We pretend that quantum points (i.e. Lagrangian submanifolds) are ordinary points in an infinite-dimensional configuration space. Then, as in the quantization of a field theory, we proceed as if this configuration space were finite dimensional. The "manifold" Q_A is a configuration space so the phase space should be the cotangent bundle T^*Q_A (see Section 2.3). It is interesting to note that the cotangent bundle over the "manifold" of all Lagrangian submanifolds has already been considered in the context of geometric quantization in order to explain certain physical phenomena, see [26]. In our case however the configuration space Q_A consists not of all Lagrangian submanifolds of M_A, but only of those which are states of A, that is of those which are tangent to the corresponding field. In the case when A is itself a coherent cotangent bundle T^*Q over some manifold Q the only closed and connected states of A are fibers of T^*Q therefore the points of Q_A are parametrized by the points of the base Q (modulo phase coordinates). Modulo phase, the coherent phase space $!A$ coincides with $T^*Q = A$ as expected.

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